Analytical Solution to the Circularity Problem in the Discounted Cash Flow Valuation Framework

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Abstract

In this paper we propose an analytical solution to the circularity problem between value and cost of capital. Our solution is derived starting from a central principle of finance that relates value today to value, cash flow, and the discount rate for next period. We derive a general formulation for the equity value, E, for the cost of unlevered equity, firm value and the weighted average cost of capital, WACC, without circularity.

We furthermore compare the results obtained using these formulas with the results using the Adjusted Present Value approach (no circularity) and the iterative solution of circularity based upon the iteration feature of a spreadsheet.

We conclude that all methods produce the same answer.

KEY WORDS

Firm valuation, cost of capital, cash flows, free cash flow, capital cash flow, WACC, circularity

JEL CLASSIFICATION

M21, M40, M46, M41, G12, G31, J33

Introduction

Since the Modigliani and Miller 1958 seminal paper, there has been a problem posed by the fact that the discount rate to value cash flows depends on the value of these cash flows. This gives raise to the circularity problem.

This problem has been addressed in different ways: ignoring it and assuming a constant cost of capital, assuming that taxes do not exist and discounting the cash flows with the cost of capital before taxes, iterating manually assuming a target leverage, and iterating automatically using the iteration feature of spreadsheets.

In this paper we propose an analytical solution to this circularity problem. Our solution is derived starting from a basic tenet of finance as follows:

$$V_t = \frac{V_{t+1} + CF_{t+1}}{1 + DR_{t+1}} \qquad (1)$$

where V is value, CF is cash flow and DR is discount rate.

We derive a general formulation for the equity value, E, at a given period and propose a general formula that depends on the value of equity and cash flow to equity for next period, the values of debt and tax savings TS at actual period, the discount rate for the TS in the next period, ψ, the cost of debt, Kd, and the unlevered cost of equity, Ku. We then present this formula for two special cases: one for ψ equal to Kd and another for ψ equal to Ku.

The remains of the paper is organized as follows: in Section One review the relevant literature on circularity; in Section Two we make a digression on target leverage; I Section Three, we present the solution to circularity and finally in Section five, we conclude.

Ella Sección Uno recoge lo que hay en la literatura al respecto de la circularidad y modo de abordarla; en la Sección Dos, se comenta sobre el endeudamiento objetivo; en la Sección Tres planteamos la solución a la circularidad; en la Sección Cuatro presentamos un ejemplo y en la Sección Cinco concluimos.

Section One. Literature Review

Authors, practitioners and teachers recognize the existence of the circularity problem and their proposed solutions range from iterative processes either manual ("rolling WACC") or automated (using a spreadsheet), to using a target leverage and assuming constant WACC[1]. Other authors such as Benninga (2006) and Benninga and Sarig (1997) simply ignore the circularity problem and just use a constant WACC or Ke, under the assumption that personal taxes approximately offset the tax shields from corporate taxes.

Authors such as Lerner and Carleton 1966, Baginski and Wahlen 2003, Pfeiffer 2004, Rao and Stevens 2007, Vishwanath 2007, Apreda 2008, Woolley 2009, and some practitioners recognize the existence of circularity but do not offer a solution to the problem.

Rao and Stevens recognize the existence of such circularity and state that "prior research has noted, but not modeled these interactions." Rao and Stevens, 2007, p.2.

Vishwanath recognizes that using book value and market values when introducing the leverage in the WACC yield different results. "The market value of equity is the present value of equity cash flows but the discount rate used to discount ECFs itself is supposed to be based on the market value of equity. That is, there is a circularity problem. We can get over this problem by using the quasi market valuation." Vishwanath, 2007. p. 559.

Even practitioners recognize the circularity problem: "Now, to be able to calculate WACC we need to know the value of the company, but to calculate that value we need to know WACC. So we have a circularity problem involving the simultaneous solution of WACC and company value." (Strategy @ Risk, Visited March 19, 2010).

There are different approaches for the solution of circularity: target capital structure, iteration by hand or rolling WACC and automatic iteration using spreadsheets.

Among the ones that propose the target capital structure and/or the iterative solution departing from the initial target leverage, we find Rosenberg and Guy, 1976; Greenwald, 1980; Luehrman, 1997 (as an introduction to his defense of the Adjusted Present Value, APV); Abarbanell, 1999; Copeland, Koller and Murrin, 2000; Abrams, 2001; Pratt , 2002; Brealey and Myers 2003; Hitchner, 2003; Schiefner, and Schmidt, 2003; Schuster and

[1] Constant leverage does not grant constant levered cost of equity, Ke, and WACC as both depend on the value of TS. See Vélez-Pareja, Ibragimov and Tham, 2008.

Jameson, 2003; Froidevaux, 2004; Schultze, 2004; Tham and Vélez-Pareja, 2004; Wood and Leitch, 2004; Mello-e-Souza and Bee, 2005; Hua and Upneja, 2005; Koller, Goedhart and Wessels. 2005; Damodaran, n.d. slides, 2000, 2006; DeMario and Fazzone, 2006; Lazar and Prisman, 2006; Vélez-Pareja, 2006; Mohanty, 2007; Penman, 2007 (cited by Liu, 2009); Crundwell, 2008; Mian and Vélez-Pareja, 2008; Pratt, 2008; Pratt and Grabowski, 2008; Turner, 2008; Ansay, 2009; Berk and Demarzo, 2009; Hess, Homburg, Lorenz, and Sievers, 2009; Liu, 2009; Lobe, 2009; Vélez-Pareja and Tham, 2000, 2005, 2009; Vélez-Pareja and Burbano-Perez, 2010; Fairchild, n.d.; Pinteris, n.d.; Mathiesen, n.d.; Tijdhof, n.d.; and the Center for Financial Research.

According to Crundwell "The values for debt and equity used in calculation of the WACC must be market values (not historical values) and they must be targeted values … not current values. This circular argument creates difficulties" (Crundwell, 2008. p. 378.).

Koller, Goedhart and Wessels are straightforward: "To value the company, use target weights." However, at the same time, they argue that "you must determine equity value (for the cost of capital) either using a multiples approach or through DCF iteratively. To perform an iterative valuation, assume a reasonable capital structure, and value the enterprise using DCF. Using the estimate of debt to enterprise value, repeat the valuation. Continue this process until the valuation no longer materially changes." (Koller, Goedhart and Wessels, 2005. p. 324-325).

Pinteris states: "Note that the choice of a target capital structure is also dictated by the presence of a circularity problem in our calculations. In order to estimate the WACC we need to know the market weights of debt and equity. In order to do so, we need to know, in particular, the market value of equity. But, this depends on the discount rate used to discount future free cash flows, which is given by the WACC. Estimating the target capital structure, we could use the current market-based capital structure of the company and review the capital structure of similar companies, as well as examine the management's policy towards financing." (Pinteris, n.d., p. 5).

Berk and Demarzo, 2009, recognize that when leverage changes, the WACC changes, and it is difficult to calculate the value; to solve this they calculate the value with the APV and then they calculate the WACC. Then they use this WACC to calculate value with the FCF and obviously, they get the same value. As it has been said, APV is the

easiest way to solve circularity. Why is it necessary to calculate WACC? Although it is not exactly the case, we recognize that the application of the formula for WACC using the results of a first method breaks the circularity.

In an introduction to his defense of APV, Luehrman says: "One expedient is to guess at the market value or use book values and then iterate –fill in the computer market value as the new guess, then recompute another guess, and so forth until the guess and the computed values converge." (Luehrman, 1997, p. 153). This is also known as rolling WACC.

Copeland, Koller and Murrin, 2000. p. 204, consider that "the second reason for using a target capital structure is that it solves the problem of circularity involved in estimating the WACC."

Brealey and Myers 2003, p. 227 and p. 25, avoid the issue of circularity assuming that they have a balance sheet with market values. If that is the situation, then they don't need to calculate WACC. They even talk of an industry cost of capital, (p. 550), but this doesn't resolve the problem either.

Professor Abarbanell, 1999, p. 6 warns the reader: "[…] plugging the actual market value of the firm into the calculation of WACC involves circular reasoning (since we are trying to determine what that market value should be!). Thus, it is necessary to guess at the firm's market value, use the guess to determine the weights to apply in the WACC and determine if the estimated WACC leads to a projected equity value of the firm equal to your original guess."

According to Lazar and Prisman "This introduces circularity into the process as if the market value of the debt and equity are known so is the value of the firm, but the value of the firm is what we try to estimate. Even in valuing a firm practitioners use book values as a solution to this problem even though it can be solved numerically. A few iterations can obtain a value of equity and debt that is consistent (to a tolerance) with the value of the firm." (Lazar and Prisman, 2006. p. 24).

Greenwald comments on the difference when appraising regulatory projects mentioning that "the basic difficulty in valuing a utility's assets (i.e., its rate base) is one of circularity. Their value is determined, like those of any asset, by the net income they are capable of producing. But, this in turn is determined by the policies of the relevant

regulatory agency and, in particular, by the value such an agency places on the assets of the utility. Thus, valuations by a regulatory authority tend to be self-fulfilling and there is no firmly based principle by means of which this circle can be broken. Attempts to break it have traditionally taken two directions." (Greenwald, 1980, p. 2). A similar problem was posed and solved by Vélez-Pareja, 2006.

Wood and Leitch say "There is no general analytical solution to this circularity, so the ordinary weighted average cost of capital cannot capture the effects of changing capital structure on the cost of capital, and the computed NPV is not correct: the wealth of the shareholders will change by a different amount, and may have a different sign as well." (Wood and Leitch, 2004, p. 16). They also state that "Such circularity precludes a general analytical solution to the problem of determining the appropriate discount rate to use for a proposed project. The FPV solution technique uses an iterative method to attack this circularity." (Wood and Leitch, 2004, p. 19)

Damodaran, 2000, recognizes that "every textbook is categorical that the weights in the cost of capital calculation be market value weights" and that the problem is the "inconsistency" behind this. To solve this inconsistency he proposes an iterative procedure. This is the "rolling WACC" that eventually "will converge sooner rather than later".

Pratt comments that "in computing WACC for a closely held company, project, or proposed project, one important additional problem exists: Because there is no market for the securities, we have to estimate market values in order to compute the capital structure weightings. As we will see, estimating the weightings for each component of the capital structure becomes an iterative process for companies intending or assumed to operate with current levels of debt. Fortunately, computers perform this exercise very quickly. (To "iterate" means to repeat. An "iterative process" is a repetitious one. In this case, we estimate market value weights because the actual market values are unknown. We may reestimate weights several times until the computed market value weights come fairly close to the weights used in estimating the WACC.)" (Pratt , 2002. p48-49).

Vélez-Pareja and Tham, 2000, 2009; Tham and Vélez-Pareja, 2004, and Vélez-Pareja and Burbano-Perez, 2010, have proposed the solution of circularity constructing the circular relation and iterating using the spreadsheet ability to handle such iterative process.

After recognizing the existence of circularity, Abrams, 2001. p. 180 mentions that "using an iterative approach eliminates this deficiency in both models. After determining the market value of debt, we can assume any value for equity to get our initial debt to equity ratio. We calculate the first iteration of equity value using this initial ratio. After several iterations, we eventually obtain a unique solution for equity that is consistent with the last iteration of the debt to equity ratio and is independent of our initial choice of equity."

Damodaran in one of his teaching slides recommends:

"Rather than use book value weights, you should try

• Industry average debt ratios for publicly traded firms in the business

• Target debt ratio (if management has such a target)

• Estimated value of equity and debt from valuation (through an iterative process)"

(Damodaran. Valuations. Slide 46).

Reporting the results of a survey on tools used in capital budgeting, Truong, Partington and Peat, 2008. p. 107 and 118 explain that "most respondents (84%) estimated a WACC. In computing the WACC, 60% of companies said they used target weights and 40% used current weights. In regard to the choice between market value and book value weights there was a substantial drop in the number of respondents. Those companies that responded show a nearly even balance between those who used market value weights (51%), and those who used book value weights (49%)". On the other hand, "the project cash flows are discounted at the weighted average cost of capital as computed by the company, and most companies use the same discount rate across divisions. The discount rate is assumed constant for the life of the project. The WACC is based on target weights for debt and equity".

Others use or modify a simple solution proposed by Myers, 1974, the Adjusted Present Value, APV. For instance, Luehrman, 1974 advocates for Myers APV; McDaniel, 1994 p. 147, considers that "the APV method of dealing with flotation costs by adjusting the initial investment is feasible for a general capital budgeting/financing case, because circularity can be avoided by using an algorithm that matches each project's NPV with the incremental flotation cost of the security potentially issued to finance the project. The APV method reduces the ambiguity of the stock price variable in the Gordon model. However,

without modification, the APV method may reject value-increasing strategies for those firms with promising long-range investment opportunities". On the other hand, Adserà and Viñolas, 2003, recognize the existence of circularity for perpetuities and propose a modified version of APV as the solution.

Finally, Vélez-Pareja and Benavides, 2006, present an analytical solution to the circularity that derives into the Capital Cash Flow.

Section Two. A Digression about Target Leverage

The idea of using target leverage is to elude or avoid the circularity problem, or if accompanied by an iteration process, to solve it. Those who elude the problem with the straightforward use of target leverage without any iteration imagine they are correctly avoiding the problem. In fact, when we assume a target leverage, usually considered constant, we do have circularity because the general formulation of cost of capital (be it the levered equity cost of capital Ke, or the weighted average cost of capital, WACC) depends on the tax savings and/or their market value. Hence, we need to calculate debt in period t-1 for the cost of capital in t and from there to the end of the planning horizon. The current practice dismisses this situation and applies the standard textbook formula as if it could be done without the rebalancing of debt and its consequence in the tax savings. (See Tham and Vélez-Pareja, 2004 and Taggart, 1991).

When the rebalancing of debt is not done, the cash flow to equity, CFE, cannot be calculated (assuming correctly that CFE is what the shareholder effectively receives). See Magni and Vélez-Pareja, 2009.

Section Three. The Solution to Circularity

Using the basic tenet of finance and the derivation for the levered cost of equity by Taggart, 1991 and Tham and Vélez-Pareja, 2004, we analytically solve the problem of circularity between the capital structure and the required rate of return. We assume that debt schedule is known from the beginning and it could have any kind of profile. A "known" debt schedule is the result of solving the needs of cash when short and long term deficits are modeled in a financial planning model; see Vélez-Pareja, 2009. These formulae are derived in Appendix A.

A general formula for any discount rate for TS, ψ is

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t).D_{t-1} + (Ku_t - \psi_t).V_{t-1}^{TS}}{1 + Ku_t} \quad (2)$$

and

$$V_{t-1} = \frac{V_t + FCF_t + TS_t + (Ku_t - \psi_t)V_{i-1}^{TS}}{1 + Ku_t} \quad (3)$$

Where E is the market value of equity, CFE is the cash flow to equity, Kd is the cost of debt, Ku is the unlevered cost of equity, D is market value of debt, ψ is the discount rate of the tax savings, TS, V is the market value of the firm, V^{TS} is the market value of TS, and FCF is free cash flow.

The formulation for $\psi = Kd$

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t)(D_{t-1} - V_{t-1}^{TS})}{1 + Ku_t} \quad (4)$$

$$V_{t-1} = \frac{V_t + FCF_t + TS_t + (Ku_t - Kd_t)V_{t-1}^{TS}}{1 + Ku_t} \quad (5)$$

The formulation for $\psi = Ku$

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t).D_{t-1}}{1 + Ku_t} \quad (6)$$

$$V_{t-1} = \frac{V_t + FCF_t + TS_t}{1 + Ku_t} \quad (7)$$

Observe that (7) is the value calculated with the capital cash flow, CCF, proposed by Ruback, 2002, and it is a basic tenet of finance, as mentioned in (1).

For WACC

$$WACC_t = \frac{Ku_t(V_t + FCF_t) - (Ku_t - \psi_t)V_{t-1}^{TS} - TS_t}{V_t + FCF_t + (Ku_t - \psi_t).V_{t-1}^{TS} + TS_t} \quad (8)$$

For $\psi = Kd$

$$WACC_t = \frac{Ku_t(V_t + FCF_t) - (Ku_t - Kd_t)V_{t-1}^{TS} - TS_t}{V_t + FCF_t + (Ku_t - Kd_t).V_{t-1}^{TS} + TS_t} \quad (9)$$

For $\psi = Ku$

$$WACC_t = \frac{Ku_t(V_t + FCF_t) - TS_t}{V_t + FCF_t + TS_t} \quad (10)$$

Tham and Vélez-Pareja, 2004, propose a calculation for the terminal value that solves the circularity. Terminal value for tax savings, V^{TV_TS} in N assuming ψ=Ku is

$$V_N^{TV\_TS} = \frac{TKdD\%\ V_N^{TV\_L}}{Ku - g} \quad (11)$$

where T is corporate tax rate, Kd is cost of debt, D% is leverage, Ku is unlevered cost of equity and g is nominal growth (all of these variables are at perpetuity) and $V^{TV\_L}{}_N$ is levered firm terminal value.

The unlevered TV

$$V_N^{TV\_Un} = V_N^{TV\_L}\left(1 - \frac{TKdD\%}{Ku - g}\right) \quad (12)$$

Solving for the levered terminal value we have

$$V_N^{TV\_L} = \frac{FCF_{N+1}}{(Ku - g)\phi} \quad (13)$$

Where FCFN+1 is the free cash flow at N+1 and ϕ is

$$\phi = 1 - \frac{TKdD\%}{Ku - g} \quad (14)$$

With this collection of formulae we solve analytically the circularity problem.

Section Four. An Example

In this example we assume ψ = Ku. In Appendix B we repeat it for ψ = Kd. In table 1 we show the input data.

Table 1a. Input data for the example

Year		1	2	3	4
CFD		23.48	13.71	14.43	17.99
TS		4.22	3.56	3.40	3.06
FCF		19.26	18.34	23.67	31.81
CFE		0.00	8.18	12.64	16.88
D	91.97	80.56	77.00	72.28	63.04
$TV_E=TV_{FCF}$ -D					184.74

Terminal Value for the firm and for the TS has been calculated using (11) and (13).

Table 1b. Input data for perpetuity and TV calculation

T	35.00%
Kd	12.10%
D%	25.44%
Ku	13.92%
g	0%
ϕ	92.26%
FCF_{N+1}	31.81
TV_{FCF}	247.78

En la tabla 2 mostramos algunos resultados para la perpetuidad o valor terminal. Usamos las ecuaciones (12) y (13).

Tabla 2. Cálculo de los valores terminales

Año	0	1	2	3	4
VT (AI) (ec. (12))					19,19
VT(FCL) (ec. (13))					247,78
VT(P)=TV(FCL)-D					184,74
VT desapalancado VT(FCL) – VT(AI)					228,6

El Valor terminal para la firma y para el AI ha sido calculado usando (12) y (13)

In table 2 we show some results for the perpetuity or terminal value. We use equations (12) and (13).

Table 2. Calculating Terminal Values

Year	0	1	2	3	4
TV(TS) (eq. (12))					19.19
TV(FCF) (eq. (13))					247.78
TV(E)=TV(FCF)-D					184.74
TV unlevered TV(FCF) – TV(TS)					228.60

Using equation (6)

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t).D_{t-1}}{1 + Ku_t} \tag{6}$$

Table 3. Calculation of Market Value of Equity using (5) and Firm Value

Year		1	2	3	4
CFE		0.00	8.18	12.64	16.88
Kd		13.12%	12.61%	12.61%	12.10%
Ku		15.00%	14.46%	14.46%	13.92%
E	127.75	148.64	163.44	175.85	184.74
D	91.97	80.56	77.00	72.28	63.04
V = D + E	219.72	229.20	240.44	248.13	247.78

Using equations (7) and (8)

$$V_{t-1} = \frac{V_t + FCF_t + TS_t}{1 + Ku_t} \tag{7}$$

In table 3 we calculate firm value using the previous equation and from it, we calculate market value of equity.

Table 4. Calculating Firm Value using (6) and Equity Market Value.

Year	0	1	2	3	4
TS		4.22	3.56	3.40	3.06
FCF		19.26	18.34	23.67	31.81
Ku		15.00%	14.46%	14.46%	13.92%
V	219.72	229.20	240.44	248.13	247.78
D	91.97	80.56	77.00	72.28	63.04
E=V-D	127.75	148.64	163.44	175.85	184.74

As expected, the two values are identical. Using (8) to estimate WACC without circularity, we have

$$WACC_t = \frac{Ku_t.(V_t + FCF_t) - TS_t}{V_t + FCF_t + TS_t} \qquad (8)$$

In table 5 we calculate firm value using FCF and WACC, equation (8).

Table 5. Calculation of Firm Value using FCF and WACC

Year	0	1	2	3	4
TS		4.22	3.56	3.40	3.06
FCF		19.26	18.34	23.67	31.81
WACC		13.08%	12.91%	13.04%	12.68%
V	219.72	229.20	240.44	248.13	247.78
D	91.97	80.56	77.00	72.28	63.04
E=V-D	127.75	148.64	163.44	175.85	184.74

Again, as expected, firm and equity values are identical to the ones found in precious approaches.

As APV is the simplest way to calculate value without circularity, we check our results with the APV and $\psi = Ku$, in table 5.

Table 6. Calculating Firm Value and Market Equity Value using APV.

Year	0	1	2	3	4
FCF		19.26	18.34	23.67	31.81
TV unlevered for FCF					228.60
TS		4.22	3.56	3.40	3.06
TV TS					19.19
PV(FCF at Ku)	198.13	208.59	220.41	228.60	228.60
PV(TS at Ku)	21.59	20.61	20.03	19.53	19.19
V	219.72	229.20	240.44	248.13	247.78
D	91.97	80.56	77.00	72.28	63.04
V-D	127.75	148.64	163.44	175.85	184.74

Results from table 6 show that the proposed analytical method gives consistent results with the APV, which is the easiest way to calculate value without the circularity problem.

Concluding Remarks

We have shown three analytical solutions for the circularity problem, namely, the calculation of equity market value, the total firm value with the WACC without circularity,. We have also shown that the solution (valuation) using the proposed methods is consistent, given an assumption on the discount rate for the TS. All three methods coincide with the APV, which is the best method to calculate value without circularity. These methods do not require neither target leverage nor iterations.

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Appendix A

Fundamental and independent equations

$$E_t = E_{t-1}.(1+Ke_t) - CFE_t \quad (A1)$$

$$Ke_t = Ku_t + (Ku_t - Kd_t).\frac{D_{t-1}}{E_{t-1}} - (Ku_t - \psi_t).\frac{V^{TS}_{t-1}}{E_{t-1}} \quad (A2)$$

$$V_t = E_t + D_t = (E_{t-1} + D_{t-1}).(1+WACC_t) - FCF_t \quad (A3)$$

From Taggart, 1991 and Tham and Vélez-Pareja, 2004, we have

$$WACC_t = Ku_t - (Ku_t - \psi_t).\frac{V^{TS}_{t-1}}{E_{t-1}+D_{t-1}} - \frac{TS_t}{E_{t-1}+D_{t-1}} \quad (A4)$$

General expression for E for any ψ

From A1, we have

$$1 + Ke_t = \frac{E_t + CFE_t}{E_{t-1}} \quad (A6)$$

Replacing (A2) in (A6)

$$1 + Ku_t + (Ku_t - Kd_t).\frac{D_{t-1}}{E_{t-1}} - (Ku_t - \psi_t).\frac{V^{TS}_{t-1}}{E_{t-1}} = \frac{E_t + CFE_t}{E_{t-1}} \quad (A7)$$

$$1 + Ku_t = \frac{E_t + CFE_t - (Ku_t - Kd_t).D_{t-1} + (Ku_t - \psi_t).V^{TS}_{t-1}}{E_{t-1}} \quad (A8)$$

Simplifying and comparing with A1, we solve for E, the market value of equity.

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t).D_{t-1} + (Ku_t - \psi_t).V^{TS}_{t-1}}{1+Ku_t} \quad (A9)$$

Equity value when ψ = Ku:

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t).D_{t-1}}{1+Ku_t} \quad (A10)$$

Formulation for E when ψ = Kd:

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t)(D_{t-1} - V^{TS}_{t-1})}{1+Ku_t} \quad (A11)$$

Derivation for V and WACC.

General expression for V = D+E for any ψ

$$E_t + D_t = (E_{t-1} + D_{t-1}).(1+WACC_t) - FCF_t \quad (A12)$$

$$E_{t-1} + D_{t-1} = \frac{E_t + D_t + FCF_t}{1 + WACC_t} \quad (A13)$$

Replacing (A13) into (A4) we have

$$WACC_t = Ku_t - (Ku_t - \psi_t).\frac{V_{t-1}^{TS}(1 + WACC_t)}{E_t + D_t + FCF_t} - \frac{TS_t(1 + WACC_t)}{E_t + D_t + FCF_t} \quad (A14)$$

Reorganizing (A15)

$$1 + WACC_t = 1 + \frac{Ku_t(E_t + D_t + FCF_t) - (Ku_t - \psi_t)V_{t-1}^{TS}(1 + WACC_t) - TS_t(1 + WACC_t)}{E_t + D_t + FCF_t} \quad (A15)$$

Simplifying

$$1 + WACC_t = \frac{(1 + Ku_t)(E_t + D_t + FCF_t) - (1 + WACC_t)[(Ku_t - \psi_t)V_{t-1}^{TS} + TS_t]}{E_t + D_t + FCF_t} \quad (A16)$$

Solving for 1+WACC:

$$1 + WACC_t = \frac{(1 + Ku_t)(E_t + D_t + FCF_t)}{E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t} \quad (A17)$$

Solving A17 for WACC:

$$WACC_t = \frac{(1 + Ku_t)(E_t + D_t + FCF_t) - [E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t]}{E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t} \quad (A18)$$

$$WACC_t = \frac{(1 + Ku_t - 1)(E_t + D_t + FCF_t) - (Ku_t - \psi_t)V_{t-1}^{TS} - TS_t}{E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t} \quad (A19)$$

$$WACC_t = \frac{Ku_t(E_t + D_t + FCF_t) - (Ku_t - \psi_t)V_{t-1}^{TS} - TS_t}{E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t} \quad (A20a)$$

$$WACC_t = \frac{Ku_t(V_t + FCF_t) - (Ku_t - \psi_t)V_{t-1}^{TS} - TS_t}{V_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t} \quad (A20b)$$

This is the general formulation for WACC without circularity.

The derivation of value V is, replacing (A18) in (A13)

$$E_{t-1} + D_{t-1} = \frac{E_t + D_t + FCF_t}{\frac{(1+Ku_t)(E_t + D_t + FCF_t)}{E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t}} \quad \text{(A21)}$$

Simplifying

$$E_{t-1} + D_{t-1} = \frac{E_t + D_t + FCF_t + (Ku_t - \psi_t)V_{t-1}^{TS} + TS_t}{1 + Ku_t} \quad \text{(A22a)}$$

$$V_{t-1} = \frac{V_t + FCF_t + TS_t + (Ku_t - \psi_t)V_{t-1}^{TS}}{1 + Ku_t} \quad \text{(A22b)}$$

Formula when ψ = Ku:

From (A23b)

$$V_{t-1} = \frac{V_t + FCF_t + TS_t}{1 + Ku_t} \quad \text{(A23)}$$

This is the basic tenet of finance applied to the Capital Cash Flow

Formulas when ψ = Kd:

From (A23)

$$V_{t-1} = \frac{V_t + FCF_t + TS_t + (Ku_t - Kd_t)V_{t-1}^{TS}}{1 + Ku_t} \quad \text{(A24)}$$

Appendix B

Example assuming ψ = Kd. In table B1 we present the input data for the example.

Table B1. Input Data for Example

Year	0	1	2	3	4
CFD		23.5	13.7	14.4	18.0
TS		4.22	3.56	3.40	3.06
FCF		19.26	18.34	23.67	31.81
CFE = = FCF + TS - CFD		0.00	8.18	12.64	16.88
TV for FCF					247.78
D	91.97	80.56	77.00	72.28	63.04
TV for E=TV FCF –D					184.74

Using equation (A11)

$$E_{t-1} = \frac{E_t + CFE_t - (Ku_t - Kd_t)(D_{t-1} - V_{t-1}^{TS})}{1 + Ku_t} \qquad (A11)$$

In table B2 we calculate market equity value directly with (A11)

Table B2. Calculating Market Value of Equity using (A11)

Year	0	1	2	3	4
CFE		0.00	8.18	12.64	16.88
Kd		13.12%	12.61%	12.61%	12.10%
Ku		15.00%	14.46%	14.46%	13.92%
D	91.97	80.56	77.00	72.28	63.04
PV(TS @ Kd)	22.73	21.49	20.64	19.85	19.19
E	128.88	149.52	164.04	176.16	184.74
V = D+E	220.86	230.07	241.05	248.44	247.78

Using eq. (A24)

$$V_{t-1} = \frac{V_t + FCF_t + TS_t + (Ku_t - Kd_t)V_{t-1}^{TS}}{1 + Ku_t} \qquad (A24)$$

Table B3 shows the calculation of firm value directly using (A24).

Table B3. Calculating Firm Value with (A24) and Market Equity Value

Year	0	1	2	3	4
TS		4.22	3.56	3.40	3.06
FCF		19.26	18.34	23.67	31.81
Ku		15.00%	14.46%	14.46%	13.92%
V	220.86	230.07	241.05	248.44	247.78
D	91.97	80.56	77.00	72.28	63.04
E=V-D	128.88	149.52	164.04	176.16	184.74

Using WACC (eq. (A20b))

$$WACC_t = \frac{Ku_t(V_t + FCF_t) - (Ku_t - Kd_t)V_{t-1}^{TS} - TS_t}{V_t + FCF_t + (Ku_t - Kd_t)V_{t-1}^{TS} + TS_t} \qquad (A20b)$$

Using (A24) and the FCF we calculate firm value and equity value in table B4.

Table B4. Calculating WACC and Firm Value

Year	0	1	2	3	4
TS		4.22	3.56	3.40	3.06
FCF		19.26	18.34	23.67	31.81
WACC		12.89%	12.74%	12.89%	12.54%
V	220.86	230.07	241.05	248.44	247.78
D	91.97	80.56	77.00	72.28	63.04
E=V-D	128.88	149.52	164.04	176.16	184.74

Again, all values coincide. As APV is the simplest way to calculate values without circularity, we show its calculations in Table B5. As expected, all previous calculations coincide with APV.

Table B5. Using APV with $\psi = Kd$

Year	0	1	2	3	4
FCF		19.26	18.34	23.67	31.81
TV unlevered for FCF					228.60
TS		4.22	3.56	3.40	3.06
TV TS					19.19
PV(FCF at Ku)	198.13	208.59	220.41	228.60	228.60
PV(TS at Kd)	22.73	21.49	20.64	19.85	19.19
V	220.86	230.07	241.05	248.44	247.78
D	91.97	80.56	77.00	72.28	63.04
V-D	128.88	149.52	164.04	176.16	184.74