Analytical Solution for Optimal Capital Structure in Perpetuities

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Abstract

We derive and present the formula for optimal debt under the assumption that tax shields are discounted at the cost of levered equity, Ke and cash flows are on perpetuity. The formulation is consistent and is derived from basic financial principles. This formulation is valid for non-growing perpetuities.

Key Words

Firm valuation, optimal capital structure, discount rate for tax shields.

JEL codes

M21, M40, M46, M41, G12, G31, J33

Introduction

Tham and Vélez-Pareja (2010) and Kolari and Vélez-Pareja (2010) present the formulation for the cost of levered equity in a world where the discount rate for tax shields is the same cost of levered equity. This formulation is

$$Ke=Ku+\frac{Ku\text{-}Kd\ \ D_{t-1}}{V^{Un}_{t-1}\text{-}D_{t-1}} \qquad (1)$$

where Ke is the levered cost of equity, Ku is the unlevered cost of equity, Kd is the cost of debt, D is debt and V^{Un} is the unlevered value of the perpetuity. Equation (1) is valid for non-growing perpetuities and for finite cash flows.

Kolari and Vélez-Pareja (2010) show that when using the cost of levered equity for discounting the tax shields, some bankruptcy costs are captured and hence an optimal capital structure is found. The identification of the optimal capital structure is found using tables or software to arrive to that optimum.

This paper presents the compact analytical procedure to calculate the optimal debt, D, given some input variables. The procedure is illustrated with a simple example.

The Optimal Formulation

Assuming non growing perpetuity with inputs such as free cash flow, FCF, Ku the unlevered cost of equity, Kd the cost of debt and corporate tax rate, T and assuming that the discount rate for tax shields is the levered cost of equity, Ke the optimal level of debt is derived.

Appendix shows the derivation of the expression for optimal debt, departing from the formula for Ke presented in Tham and Vélez-Pareja (2010) and Kolari and Vélez-Pareja (2010) the analytical expression for optimal D is as follows:

$$D_{Opt} = \frac{FCF\left(1 \pm \sqrt{1 - \frac{Kd}{Ku}}\right)}{Kd} \tag{2a}$$

As this equation has two roots, the solution is found with the lower one, (see Appendix) this is

$$D_{Opt} = \frac{FCF\left(1 - \sqrt{1 - \frac{Kd}{Ku}}\right)}{Kd} \tag{2b}$$

Kolari and Vélez-Pareja (2010) present a simple example where the input data is

Table 1. Input data for a non growing perpetuity

FCF	0,7
Ku	10%
Kd	4%
T	30%

With these inputs the following optimal debt and optimal D% is found.

Table 2. Optimal firm value and debt

D% = D/VL	VL	D
0%	7.000	0.000
5%	7.041	0.352
10%	7.082	0.708
15%	7.120	1.068
20%	7.156	1.431
25%	7.190	1.797
30%	7.220	2.166
35%	7.247	2.536
40%	7.269	2.908
45%	7.286	3.279
50%	7.298	3.649
54,015%	*7.303*	*3.944558288*
60%	7.302	4.381
65%	7.293	4.740
70%	7.276	5.093
75%	7.251	5.438
80%	7.217	5.774
85%	7.175	6.099
90%	7.125	6.412
95%	7.066	6.713
100%	7.000	7.000

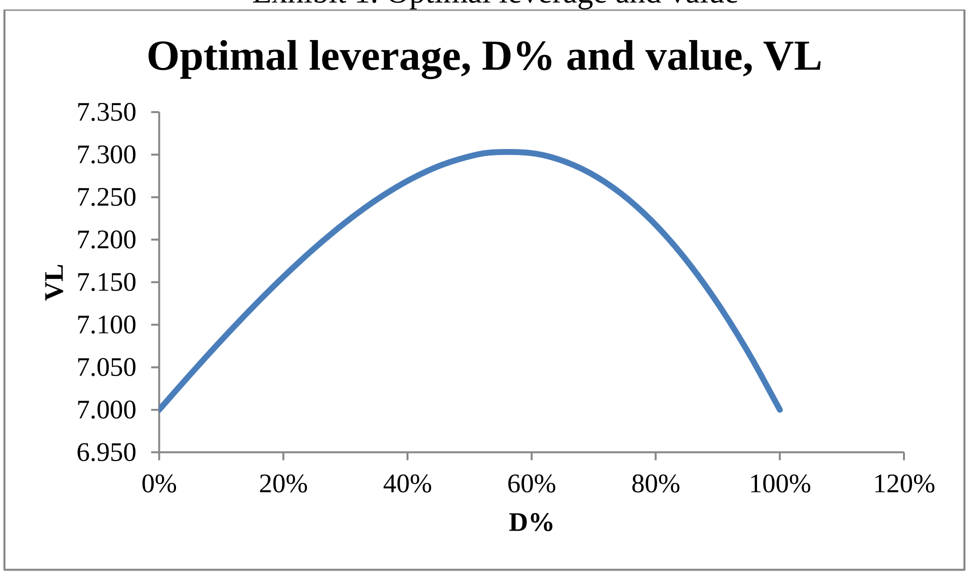
This same result can be found using (2b)

Table 3. Calculating Optimal Debt, D_{opt}

FCF(1-Kd/Ku)/Kd	
FCF	0.7
Kd/Ku	0.4
1-Kd/Ku	0.6
Root of (1-Kd/Ku)	0.774596669
1-Root of (1-Kd/Ku)	0.225403331
FCF(1-Root of (1-Kd/Ku))	0.157782332
D=FCF(1-Root of (1-Kd/Ku))/Kd	3.944558288

With this debt, VL, the levered value of perpetuity is found to be 7.303 (see Kolari and Velez-Pareja, (2010)). The optimal leverage and value is depicted in exhibit 1.

Exhibit 1. Optimal leverage and value



Concluding Remarks

This paper has shown the analytical solution of capital structure optimality when the cash flow is a perpetuity.

Further work is required to find the relationship between cost of debt and leverage. Formula (1) for Ke captures the bankruptcy costs associated with leverage except the effect of such leverage on the cost of debt.

Bibliographic References

Kolari, J. W. and Velez-Pareja, I., (2010). Corporation Income Taxes and the Cost of Capital: A Revision (November 25). Available at SSRN: http://ssrn.com/abstract=1715044

Tham, J. and Velez-Pareja, I., (2010). Cost of Capital with Levered Cost of Equity as the Risk of Tax Shields (August 9). Available at SSRN: http://ssrn.com/abstract=1655244.

Appendix

Analytical formula for optimal debt

$$V^{TS} = \frac{DKdT}{\psi} \qquad \text{(A1)}$$

Assume ψ = Ke

Then

$$Ke = Ku + \frac{Ku\text{-}Kd \;\; D}{V^{Un} - D} \qquad \text{(A2)}$$

And

$$V^{TS} = \frac{DKdT}{Ke} \qquad \text{(A3)}$$

$$\frac{\partial VTS}{\partial D} = \frac{Ke \times dT - DdT \times (\partial Ke/\partial D)}{Ke^2} \qquad \text{(A4)}$$

Set $\partial V^{TS}/\partial D$ equal to zero.

$$Ke \times KdT = DKdT \times (\partial Ke/\partial D) \qquad \text{(A5)}$$

$$Ke = D \times (\partial Ke/\partial D) \qquad \text{(A6)}$$

$$Ke = \frac{D \times V^{Un} \times (Ku - Kd)}{{V^{Un} - D}^{\;2}} \qquad \text{(A7)}$$

Reorganizing terms

$$\frac{Ke \times (V^{Un} - D)}{V^{Un}} = \frac{(Ku - Kd) \times D}{V^{Un} - D} \qquad \text{(A8)}$$

From A2, the right hand of (A8) can be written as

$$\frac{Ke \times (V^{Un} - D)}{V^{Un}} = Ke - Ku \qquad \text{(A9)}$$

Simplifying and solving for D

$$D = \frac{Ku \times V^{Un}}{Ke} \qquad \text{(A10)}$$

$$D \times Ke = FCF \qquad \text{(A11)}$$

Replacing Ke by (A2)

$$DKu + \frac{Ku - Kd\ D^2}{V^{Un} - D} = FCF \qquad (A12)$$

$$(Ku - Kd)D^2 = (FCF - DKu)\times(V^{Un} - D) \qquad (A13)$$

$$(Ku - Kd)D^2 = (FCF)\times V^{Un} - FCF\times D - DKu\times V^{Un} + KuD^2 \qquad (A14)$$

$$-KdD^2 = FCF\times V^{Un} - FCF\times D - DKu\times V^{Un} \qquad (A15)$$

$$-KdD^2 = (FCF)\times V^{Un} - 2\times FCF\times D \qquad (A16)$$

$$KdD^2 - 2\times FCF\times D + (FCF)\times V^{Un} = 0 \qquad (A17)$$

$$D = \frac{2FCF \pm \sqrt{4FCF^2 - 4FCF\ KdV^{Un}}}{2Kd} \qquad (A18)$$

$$D = \frac{FCF \pm \sqrt{FCF^2 - FCF\ KdV^{Un}}}{Kd} \qquad (A19)$$

$$D = \frac{FCF \pm \sqrt{FCF^2 - FCF^2 Kd/Ku}}{Kd}$$

$$D = \frac{FCF \pm FCF\ \sqrt{1 - Kd/Ku}}{Kd} \qquad (A20)$$

Assume the two cases and reorganize the equation as follows:

$$DKd = CFD = FCF\ \ 1 - \sqrt{1 - Kd/Ku} \qquad (A22a)$$

$$DKd = CFD = FCF\ \ 1 + \sqrt{1 - Kd/Ku} \qquad (A22b)$$

In the first case, CFD < FCF; in the second case, CFD > FCF. By definition, CFD cannot be greater than FCF.

The proper solution is

$$D = \frac{FCF - FCF\ \sqrt{1 - Kd/Ku}}{Kd} \qquad (A22a)$$