Optimal Capital Structure for Finite Cash Flows

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First version: December 27, 2010
This version: March 31, 2011

Abstract

This paper shows how to proceed to find the optimal capital structure and value with period-to-period constant and variable leverage, when the discount rate for tax shields is Ke, the cost of levered equity. Numerical procedures and recursive closed-form non-circular expressions for the finite-period and perpetuity cases are presented, which facilitate any kind of implementation including Montecarlo simulations.

Keywords

Optimal capital structure, valuation, non-circularity, finite cash flows, perpetuities tax shield, cost of equity

JEL Classification

M21, M40, M46, M41, G12, G31, J33

1. Introduction

Tham, Velez-Pareja and Kolari (2010) derived the formula for the cost of equity when the discount rate for the tax shields, TS, is the levered cost of equity and showed that the formula works for perpetuities and finite cash flows. This formula was later used by Kolari and Velez-Pareja, (2010) where they show that with this framework, an optimal capital structure for perpetuities without growth is found.

This paper shows how to find the optimal leverage for finite cash flows in two cases: with a unique and constant leverage and with varying leverage for each period; in addition, a general formula for perpetuities with constant growth is presented. The paper also contains simple numerical examples to illustrate the procedure. Each case tests for consistency. In both cases the discount rate for the tax shields is the cost of levered equity, Ke as proposed in Tham, Velez-Pareja, and Kolari (2010) and Kolari and Velez-Pareja, (2010). The formula for Ke that will be extensively used in this work is

$$\mathbf{Ke_t = \psi_t = Ku_t + \frac{(Ku_t - Kd_t).D_{t-1}}{VU_{t-1} - D_{t-1}}} \qquad (1)$$

In the previous formula ψ_t stands for the discount rate of the tax shield, Ku_t corresponds to the unlevered cost of equity, D_{t-1} equals the debt level, and the value of the unlevered company is denoted by VU_{t-1}. Note that the sub-indexes "t" and "t-1" are used to denote two successive periods of time.

2. The case of finite periods and constant leverage

In this case the procedure maximizes the levered value with a period-to-period constant leverage subject to the restriction that its value must be a number between 0 and 1. Thus, the optimizing model is

Max VL
Subject to
0 ≤ D% ≤ 1
VL is levered value and D% is the constant leverage

The model is constructed assuming some input variables such as corporate tax rate T, cost of debt, Kd_t, unlevered cost of equity, Ku_t, constant leverage, D% and free cash flow, FCF_t. Table 1, presented next, depicts the initial values for those variables; as said, only D% is a changing variable in the model. Other input variables are constant (for the sake of exposition clarity, input variables are shown in shaded cells).

Table 1. Input Data

Year	1	2	3	4
T	35%	35%	35%	35%
Kd	11.00%	11.00%	11.00%	11.00%
Ku	15.00%	15.00%	15.00%	15.00%
D%	50.0000%			
FCF	17.00	20.00	22.00	25.00

The model makes extensive use of the basic cash flow and value equilibrium equations for any period t, posed by Modigliani and Miller (1958) as follows:

$$\mathbf{CCF_t = FCF_t + TS_t = CFD_t + CFE_t} \qquad \mathbf{(2)}$$

Where CCF is capital cash flow, FCF is free cash flow, TS is tax shields, CFD is cash flow to debt and CFE is cash flow to equity.

$$\mathbf{VL_t = VU_t + VTS_t = E_t + D_t} \qquad \mathbf{(3)}$$

Equations (2) and (3) are used to test consistency, because compliance with them leads to a perfect matching among different methods of valuation. Recall that all valuation methods that use discounted cash flows have to provide the same answer with no rounding errors. From the input data intermediate and temporary results are calculated. These are: debt D at end of period, debt payment as $D_t - D_{t-1}$, interest charges calculated as $D_{t-1}.Kd_t$, tax shields TS as $D_{t-1}.Kd_t.T$, cash flow to debt, CFD_t, as the sum of debt payment plus interest charges, cash flow to equity solving CFE_t from (2), firm unlevered value VU as the present value of FCF at Ku, Ke according to (1), VTS as the present value of TS at Ke, and the unlevered value of equity, as VL - D - VTS. These values are depicted in Table 2. The calculation of values with the different methods below is temporary until circularity is solved and the optimizing procedure is applied.

In this case we find two stages where circularity appears: one is defining debt, D which is needed to define CFD, TS and CFE. The second stage is defining discount rates for CCF and FCF as in calculating value with WACC and FCF. Hence, the first action to be done is enabling the iteration feature in the spreadsheet.

Table 2. Intermediate and temporary values

Year	0	1	2	3	4
VU = PV(FCL a Ku)	58.66	50.46	38.03	21.74	
Debt D at end of period	30.50	26.04	19.48	11.05	-
Debt payment		4.46	6.56	8.43	11.05
Interest charges		3.36	2.86	2.14	1.22
Tax shields TS		1.17	1.00	0.75	0.43
CFD		7.82	9.43	10.58	12.26
CFE = FCF - CFD + TS		10.36	11.58	12.17	13.16
Ke = Ku + (Ku-Kd)D/(VU- D)[1]		19.33%	19.27%	19.20%	19.13%
VTS	2.34	1.62	0.93	0.36	
E- VTS= VL-D-VTS	28.16	24.42	18.55	10.69	

When debt is known, methods such as the Adjusted Present Value, APV, do not present circularity. In this case it does because debt is not known and TS depends on debt. The first method is the Adjusted Present Value, APV. In this case, the value of TS is calculated with Ke and VL is a temporary value because TS depends on D and D depends on VL.

Table 3a. Method 1: APV. Temporary Values

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
PV(FCF at Ku)	58.66	50.46	38.03	21.74	
PV(TS at Ke)	2.34	1.62	0.93	0.36	
Total value, VL	61.01	52.08	38.96	22.10	
Debt, D	30.50	26.04	19.48	11.05	

There is circularity between tables 2 and 3. Using Solver we optimize on D% and obtain $D\%_{Opt}$ = 75,2587%.

[1] In the tables that follow time sub indexes are eliminated to make it clear the reading. It is understood that TS, debt and equity values (and D% and E%) are situated in the previous period.

Exhibit 1. Dialog box from Solver when the constant leverage optimizing model is introduced

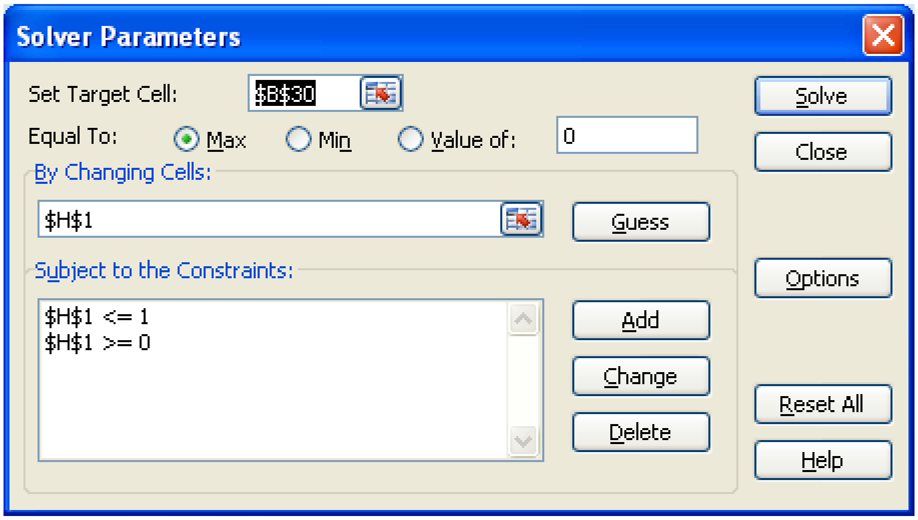


Table 3b. Optimal Value with APV

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
PV(FCF at Ku)	58.66	50.46	38.03	21.74	
PV(TS at Ke)	3.03	2.16	1.27	0.50	
Total value, VL	61.70	52.62	39.31	22.24	
Debt, D	46.43	39.60	29.58	16.74	

The second method calculates the value of levered equity and V^L is the sum of debt and equity. This method is depicted in tables 4 and 5.

Table 4. Intermediate and Temporary Values for Valuing E

Year	0	1	2	3	4
VU = PV(FCL a Ku)	58.66	50.46	38.03	21.74	
Debt D at end of period	38.29	31.16	22.27	12.07	-
Debt payment		7.13	8.89	10.20	12.07
Interest charges		4.21	3.43	2.45	1.33
Tax shields TS		1.47	1.20	0.86	0.46
CFD		11.34	12.31	12.65	13.40
CFE = FCF - CFD + TS		7.13	8.89	10.20	12.07
E- VTS= VL-D-VTS	20.37	19.31	15.76	9.67	

Table 5a. Method 2: Market Equity Value. Temporary Value (1)

Year	0	1	2	3	4
CFE		7,13	8,89	10,20	12,07
(1) Market equity value PV(CFE at Ke)	38,29	31,16	22,27	12,07	-
(2) Value of debt	38,29	31,16	22,27	12,07	
(3) Ke =Ku + (Ku-Kd).D/(VU - D)					
(4) VL = E + D	76,58	62,32	44,54	24,14	

We calculate E with an initial Ke = 0 in order to avoid a division by zero. Calculating Ke we have

Table 5b. Method 2: Market Equity Value. Temporary Value (2)

Year	0	1	2	3	4
CFE		10.36	11.58	12.17	13.16
(1) Market equity value PV(CFE at Ke)	30.50	26.04	19.48	11.05	-
(2) Value of debt	30.50	26.04	19.48	11.05	
(3) Ke =Ku + (Ku-Kd).D/(VU - D)		19.33%	19.27%	19.20%	19.13%
(4) VL = E + D	61.01	52.08	38.96	22.10	

As in method 1, table 4 depends on table 5. When Solver is used to find optimal D% we find D% = 75,2587% and values are

Table 5c. Method 2: Market Equity Value. Optimal Value

Year	0	1	2	3	4
CFE		6.85	7.15	7.04	7.06
(1) Market equity value PV(CFE at Ke)	15.26	13.02	9.72	5.50	-
(2) Value of debt	46.43	39.60	29.58	16.74	
(3) Ke =Ku + (Ku-Kd).D/(VU - D)		30.18%	29.58%	29.00%	28.39%
(4) VL = E + D	61.70	52.62	39.31	22.24	

Using the CCF and the weighted average cost of capital for the CCF, $WACC^{CCF}$, levered value is obtained. Tham and Velez-Pareja (2004) present the general formulation for $WACC^{CCF}$ as

$$\mathbf{WACC_t^{CCF} = Ku_t\text{-}(Ku_t\text{-}\psi_t).VTS_{t-1}/VL_{t-1}} \quad \textbf{(4a)}$$

When ψ, the discount rate for TS is Ke, equation (4a) becomes

$$\mathbf{WACC_t^{CCF} = Ku_t\text{-}(Ku_t\text{-}Ke_t).VTS_{t-1}/VL_{t-1}} \quad \textbf{(4b)}$$

Valuating CCF at $WACC^{CCF}$ is the third method to value the cash flows. This is depicted in Tables 6 and 7.

Table 6. Intermediate and temporary values

Year	0	1	2	3	4
VU = PV(FCL a Ku)	58.66	50.46	38.03	21.74	
Debt D at end of period	44.23	34.88	24.21	12.75	-
Debt payment		9.35	10.67	11.47	12.75
Interest charges		4.87	3.84	2.66	1.40
Tax shields TS		1.70	1.34	0.93	0.49
CFD		14.22	14.51	14.13	14.15
CFE = FCF - CFD + TS		4.49	6.83	8.80	11.34
Ke = Ku + (Ku-Kd)D_t-1/(VU_{t-1} - D_{t-1})		27.26%	23.95%	22.01%	20.67%
VTS	2.89	1.97	1.10	0.41	
VU-D	14.43	15.58	13.82	8.99	

Table 7a. Method 3: CCF and WACC^{CCF}. Temporary Values (1).

Year	0	1	2	3	4
Capital Cash Flow (CCF) = CFD + CFE		18.70	21.34	22.93	25.49
WACC^{CCF} = Ku-(Ku-Ke).VTS/VL					
PV(CCF) = VL	88.47	69.77	48.42	25.49	

Previous table is a temporary one because there is circularity between WACC^{CCF} and VL. The temporary VL is calculated with a WACC^{CCF} of zero. Introducing WACC^{CCF}, we obtain,

Table 7b. Method 3: CCF and WACC^{CCF} Temporary value (2).

Year	0	1	2	3	4
Capital Cash Flow (CCF) = CFD + CFE		18.17	21.00	22.75	25.43
WACC^{CCF} = Ku-(Ku-Ke).VTS/VL		15.17%	15.13%	15.10%	15.07%
PV(CCF) = VL	61.01	52.08	38.96	22.10	

Optimizing on D% we find D% = 75,2587% and values are

Table 7c. Method 3: CCF and WACC^{CCF}. Optimal Value.

Year	0	1	2	3	4
Capital Cash Flow (CCF) = CFD + CFE		18.79	21.52	23.14	25.64
WACC^{CCF} = Ku-(Ku-Ke).VTS/VL		15.75%	15.60%	15.45%	15.30%
PV(CCF) = VL	61.70	52.62	39.31	22.24	

The popular textbook formula for WACC for the FCF is the fourth method. As in the case of CCF, there is circularity because the calculation of D% depends on VL.

Table 8a. Method 4: Traditional Textbook WACC. Temporary Values (1).

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
(1) V^L at t=PV(FCF at WACC)	84.00	67.00	47.00	25.00	
Contribution of debt to WACC					
(2) D%		50.00%	50.00%	50.00%	50.00%
(3) Kd(1-T)		7.15%	7.15%	7.15%	7.15%
(4) Contribution KdD%(1-T)		3.58%	3.58%	3.58%	3.58%
(4a) Debt	42.00	33.50	23.50	12.50	
Contribution of equity to WACC					
(5) E%=1-D%		50.00%	50.00%	50.00%	50.00%
(6) Ke = Ku + (Ku-Kd).D/(VU - D)		25.08%	22.90%	21.47%	20.41%
(7) Contribution Ke.E% to WACC		12.54%	11.45%	10.73%	10.21%
(8) WACC = Ke.E% + Kd.(1-T).D%					

VL is a temporary value because $WACC^{FCF}$ has not been calculated because of circularity. When we introduce $WACC^{FCF}$, we find

Table 8b. Method 4: Traditional Textbook WACC. TemporaryValue (2).

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
(1) V^L at t=PV(FCF at WACC)	61.01	52.08	38.96	22.10	
Contribution of debt to WACC					
(2) D%		50.00%	50.00%	50.00%	50.00%
(3) Kd(1-T)		7.15%	7.15%	7.15%	7.15%
(4) Contribution KdD%(1-T)		3.58%	3.58%	3.58%	3.58%
(4a) Debt	30.50	26.04	19.48	11.05	
Contribution of equity to WACC					
(5) E%=1-D%		50.00%	50.00%	50.00%	50.00%
(6) Ke = Ku + (Ku-Kd).D/(VU - D)		19.33%	19.27%	19.20%	19.13%
(7) Contribution Ke.E% to WACC		9.67%	9.63%	9.60%	9.57%
(8) WACC = Ke.E% + Kd.(1-T).D%		13.24%	13.21%	13.18%	13.14%

Optimizing on D% we find $D\%_{Opt}$ = 75,2587% and values are

Table 8c. Method 4: Traditional Textbook WACC. Optimal Value.

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
(1) V^L at t=PV(FCF at WACC)	61.70	52.62	39.31	22.24	
Contribution of debt to WACC					
(2) D%		75.26%	75.26%	75.26%	75.26%
(3) Kd(1-T)		7.15%	7.15%	7.15%	7.15%
(4) Contribution KdD%(1-T)		5.38%	5.38%	5.38%	5.38%
(4a) Debt	46.43	39.60	29.58	16.74	
Contribution of equity to WACC					
(5) E%=1-D%		24.74%	24.74%	24.74%	24.74%
(6) Ke = Ku + (Ku-Kd).D/(VU - D)		30.18%	29.59%	29.00%	28.39%
(7) Contribution Ke.E% to WACC		7.47%	7.32%	7.17%	7.02%
(8) WACC = Ke.E% + Kd.(1-T).D%		12.85%	12.70%	12.56%	12.40%

Tham and Velez-Pareja (2004) present the general formulation for $WACC^{FCF}$ as

$$\mathbf{WACC_t^{FCF} = Ku_t - TS_t/VL_{t-1} - (Ku_t - \psi_t).VTS_{t-1}/VL_{t-1}} \qquad (5a)$$

When ψ is Ke then (5a) becomes

$$\mathbf{WACC_t^{FCF} = Ku_t - TS_t/VL_{t-1} - (Ku_t - Ke_t).VTS_{t-1}/VL_{t-1}} \qquad (5b)$$

Table 9a. Method 5: VL with $WACC^{FCF}$ from (5b). Temporary Values.

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
(1) Value VL	84.00	67.00	47.00	25.00	
(2) $WACC^{FCF}$ = Ku - TS/VL - (Ku-Ke).VTS/VL					

In this stage we have

Table 10. Debt and Cash Flows. Temporary Values (1).

Year	0	1	2	3	4
Debt D at end of period	42.00	26.04	19.48	11.05	-
Debt payment		15.96	6.56	8.43	11.05
Interest charges		4.62	2.86	2.14	1.22
Tax savings TS		1.62	1.00	0.75	0.43
VP(FCL a Ku)	58.66	50.46	38.03	21.74	
Ke = Ku + (Ku-Kd)D_{t-1}/(Vun_{t-1} - D_{t-1})		25.08%	19.27%	19.20%	19.13%
VTS	2.59	1.62	0.93	0.36	

Introducing $WACC^{FCF}$ with these data we have

Table 10a. Method 5: VL with $WACC^{FCF}$ from (5b) Temporary Values (2).

Year	0	1	2	3	4
FCL		17.00	20.00	22.00	25.00
(1) Value	61.01	52.08	38.96	22.10	
(2) WACC = Ku - TS_t/V_{t-1} - (Ku-Ke)$VTSt\text{-}_1/VL_{t-1}$		13.24%	13.21%	13.18%	13.14%

Now we optimize on D% and $D\%_{Opt}$ = 75,2587%.

Table 10b. Method 5: VL with $WACC^{FCF}$ from (5b). Optimal Value.

Year	0	1	2	3	4
FCL		17.00	20.00	22.00	25.00
(1) Value	61.70	52.62	39.31	22.24	
(2) WACC = Ku - TS_t/V_{t-1} - (Ku-Ke)$VTSt\text{-}_1/VL_{t-1}$		12.85%	12.70%	12.56%	12.40%

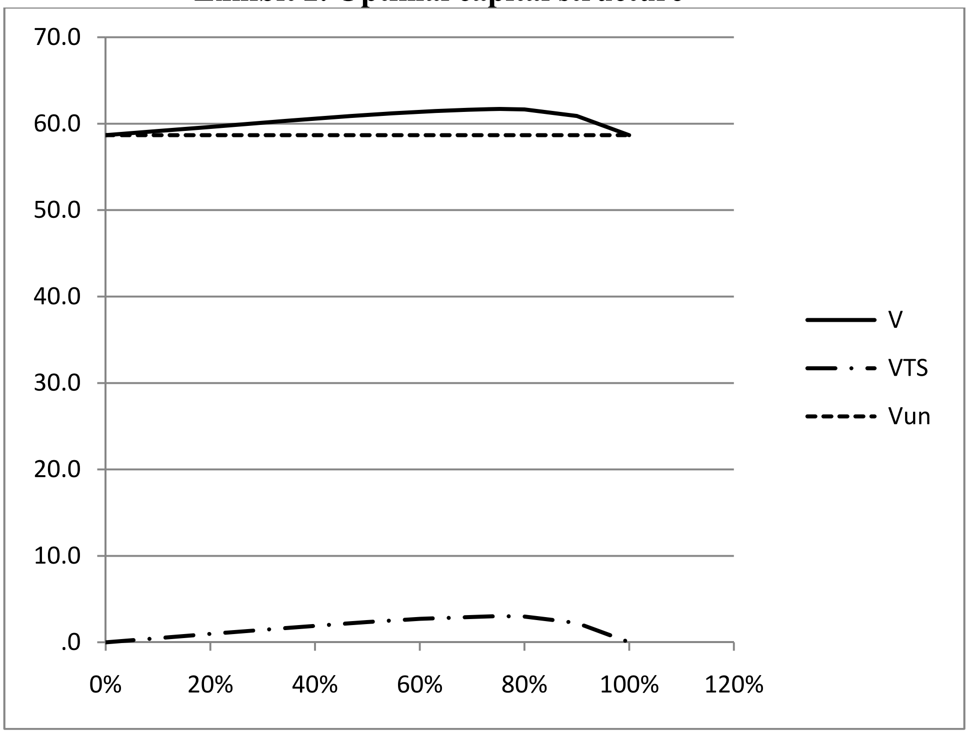
As can be seen, all methods yield the same and identical value. This is, they are consistent. Note that given a temporary leverage of 50%, all methods yield the same value, this is, 61.01. Now, using Excel Solver, the objective cell might be any of the values (in this case, the temporary one, 61.01 obtained with the APV) and set Solver to maximize that value, changing the cell where D% is written and subject to 0 ≤ D% ≤ 1. The solution by Solver is D% = 75.2587%. With this optimal D% the previous tables show the optimal values. Using one way tables one can show the behavior of value, unlevered value and VTS. In that table we observe the maximum value at D% = 75.2587%.

Table 11. Behavior of VL and VTS depending on D%

D%	VL	VTS	Vun	$E\text{-}V^{TS}$
0%	58.7	-	58.7	58.7
10%	59.2	0.5	58.7	52.7
20%	59.6	1.0	58.7	46.7
30%	60.1	1.4	58.7	40.6
40%	60.6	1.9	58.7	34.4
50%	61.0	2.3	58.7	28.2
60%	61.4	2.7	58.7	21.8
75.2587%	61.7	3.0	58.7	12.2
80%	61.6	3.0	58.7	9.4
90%	60.9	2.2	58.7	3.9
99.98%	58.7	0.0	58.7	0.0

This behavior is depicted in Exhibit 2.

Exhibit 2. Optimal capital structure



When the leverage is allowed to vary from year to year, the procedure is similar, except that when optimizing the procedure is subject to several variables (several D%, one for each year).

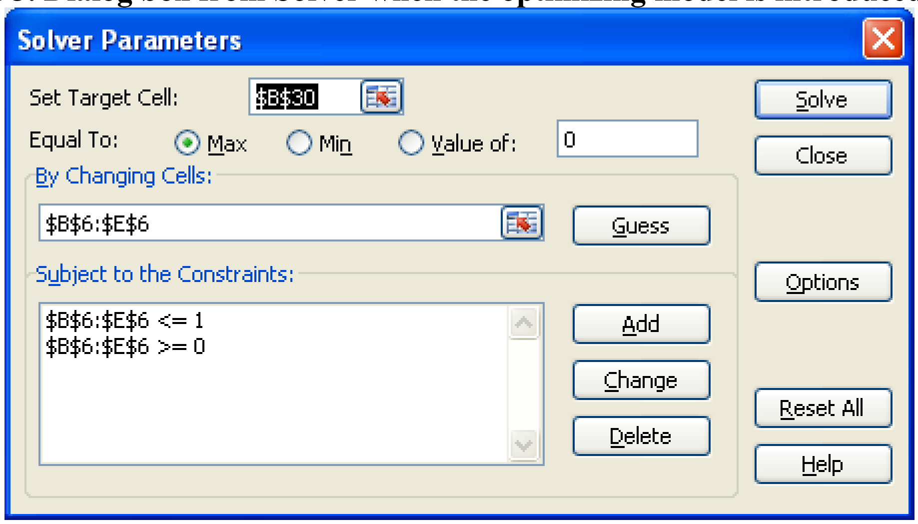
Next the reader will find the tables for a non constant leverage and maximum value. Obviously, in this case it is not possible to graph values against leverage. The inputs are identical to table 1, except that D% is variable from year 1 to year 4. In this case the procedure maximizes the levered value changing variable leverage subject to the restriction that leverage should be a value between 0 and 1.

The optimizing model is

Max VL
Subject to
$0 \leq D\%_t \leq 1$
VL is levered value and $D\%_t$ is period-to-period leverage.

Using Solver and introducing the previous mathematical model, the optimal values of D% are found.

Exhibit 3. Dialog box from Solver when the optimizing model is introduced



In this case we do not show the intermediate temporary tables. Next tables show the final result.

Table 12. Optimal values for capital structure, variable D%

Year	0	1	2	3	4
D%	72.983%	75.600%	78.566%	82.324%	
FCF		17.00	20.00	22.00	25.00
Debt D at end of period	45.04	39.79	30.90	18.32	
Debt payment		5.25	8.90	12.57	18.32
Interest charges		4.95	4.38	3.40	2.02
Tax shields TS		1.73	1.53	1.19	0.71
CFD		10.20	13.28	15.97	20.34
CFE = FCF - CFD + TS		8.53	8.26	7.22	5.37
PV(FCL a Ku)	58.66	50.46	38.03	21.74	
Ke = Ku + (Ku-Kd).D/(VU - D)		28.22%	29.92%	32.31%	36.45%
VTS	3.05	2.17	1.29	0.52	

Next tables show the optimal values obtained after using Solver for variable D%. Table 19 depicts levered value, VL calculated with CCF and $WACC^{CCF}$.

Table 13. Method 1: CCF and $WACC^{CCF}$ with optimal variable D%

Year	0	1	2	3	4
Capital Cash Flow (CCF)		18.73	21.53	23.19	25.71
$WACC^{CCF}$ = Ku-(Ku-Ke).VTS/VL		15.65%	15.62%	15.57%	15.50%
PV(CCF) = VL	61.71	52.64	39.32	22.26	

APV is the most reliable method to calculate the value of a firm. It is depicted in table 20.

Table 14. Method 2: APV with optimal variable D%

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
Tax shields TS		1.73	1.53	1.19	0.71
PV(FCF at Ku)	58.66	50.46	38.03	21.74	
PV(TS at Ke)	3.05	2.17	1.29	0.52	
Total value VL	61.71	52.64	39.32	22.26	

Method 3 is depicted in Table 15. It shows the textbook formula for $WACC^{FCF}$ and the levered value VL.

Table 15. Method 3: Textbook formula for $WACC^{FCF}$ with optimal variable D%

Year	0	1	2	3	4
FCF		17.00	20.00	22.00	25.00
(1) VL at t = PV(FCF at WACC)	61.71	52.64	39.32	22.26	
Contribution of debt to WACC					
(2) D%		72.98%	75.60%	78.57%	82.32%
(3) Kd.(1-T)		7.15%	7.15%	7.15%	7.15%
(4) Contribution Kd.D%.(1-T)		5.22%	5.41%	5.62%	5.89%
Contribution of equity to WACC					
(5) E% = 1-D%					
(6) Ke = Ku + (Ku-Kd).D/(VU - D)		27.02%	24.40%	21.43%	17.68%
(7) Contribution Ke.E% to WACC		28.22%	29.92%	32.31%	36.45%
(8) WACC = Ke.E% + Kd.(1-T).D%		7.62%	7.30%	6.93%	6.44%

Next table depicts the calculation of market levered equity with optimal leverage, D%. Table 16 includes the calculation of VL using equation (3).

Table 16. Method 4: Market equity value, with optimal variable D%

Year	0	1	2	3	4
CFE = FCF - CFD + TS		8.47	8.19	7.16	5.32
(1) Market equity value PV(CFE at Ke)	16.67	12.84	8.43	3.93	-
(2) Value of debt	45.04	39.79	30.90	18.32	
(3) Ke =Ku + (Ku-Kd)D/(V^{Un} - D)		28.22%	29.92%	32.31%	36.45%
(4) VL	61.71	52.64	39.32	22.26	

Finally, method 5 in table 17 depicts the calculation of value with the general $WACC^{FCF}$ when Ke is the discount rate for TS.

Table 17. Method 5: V^L with $WACC^{FCF}$ from (5b) with optimal variable D%

Year	0	1	2	3	4
FCL		17.00	20.00	22.00	25.00
(1) VL	61.71	52.64	39.32	22.26	
(2) WACC = Ku - TS/VL - (Ku-Ke).VTS/VL		12.84%	12.70%	12.54%	12.33%

From an Optimization Theory point of view, the restrictions present in a problem play the role of reducing the space of feasible solutions. Hence, that space is greater (or in extremely special cases, equal) in the variable leverage problem than in the constant one, which implies that in almost any case the optimized VL with variable leverage will yield better solutions than the obtained in the constant case. This is due to the fact that the space of feasible solutions in the constant case is always a sub-set of the corresponding space in the variable leverage case. Hence, the difference between the solution with constant leverage and variable leverage is as expected.

On the other hand, the analytical formulation for constant leverage is almost intractable. We consider that in the real world what happens is a variable leverage instead of a constant one, although that is a managerial decision and the constant leverage could be eventually, achieved.

3. A General Analytical Solution for Period-to-Period Variable Leverage

Next, closed-form analytical expressions for the optimal capital structure calculation in two different scenarios are presented. The first one corresponds to the finite period case and the detailed derivation of the formula can be found in Appendix A.

The problem to be solved involves finding a set of optimal levels of debt for every period that, when put together, maximize the value of the levered firm. In principle, this objective can be achieved by treating the level of debt in every single period as an unknown variable, writing down a formula for the company levered value as a function of those unknowns, finding de derivatives of that function with respect to the debt level in every period, equaling to zero every one of those derivatives, constructing a system with the equations obtained, an solving the system for the optimal debt in every period. Nonetheless, this process is extremely cumbersome and impractical, due not only to the need of solving the equation system, but also to the fact that the optimal debt level in an arbitrary period "t" is a function of the debt levels in all posterior periods; this, in turn, implies that the derivatives get more complex as the number of periods increases.

Fortunately, the last mentioned fact provides a way to find an elegant solution to the problem. First, observe that the optimal debt level in the last period "n-1" does not depend on the debt of any previous period (it is assumed that the debt level in period "n" is zero and the outstanding value in period n-1 is paid with the concomitant reduction in the CFE in period "n"); in consequence, the optimal debt level for that period can be found in an independent way. Accordingly, this optimization problem involves a single variable of choice and can be solved in a relatively straightforward way. Next, and knowing that the capital structure of period "n-1" has been optimized and correspondingly, that the value of the levered company has been maximized in that period, it is possible to find using that results (the optimal debt level for period "n-1" can be treated now as a constant) the optimal debt level for the period "n-2" in an analogous manner. This procedure can be repeated until period zero is reached, thus obtaining the optimal set of debt for every period that maximize the present value of the levered company.

In consequence, the result of this analysis (see Appendix A for details, as mentioned above) is a recursive expression that should be applied backwards in time (this is, starting from the next to last period "n-1" and using the results obtained to extend the process until period zero is reached) which, in addition, the procedure does not suffer of any circularity issues. The mentioned expression is shown next:

$$\mathbf{D_{OPT,t-1}} = \frac{\mathbf{VU_{t-1}.(1+Ku_t)}}{\mathbf{(1+Kd_t)}}.\left\{\mathbf{1}-\sqrt{\mathbf{1}-\frac{\mathbf{(1+Kd_t)}}{\mathbf{(1+Ku_t)}}.\left[\mathbf{1}-\frac{\mathbf{VTS_t.(Ku_t-Kd_t)}}{\mathbf{VU_{t-1}.(1+Ku_t).Kd_t.T}}\right]}\right\} \qquad (6)$$

This formula (6) is the same (A34) derived in Appendix A. Observe that the optimal debt level in "t-1" depends only on values in "t", which are all known. The only required value corresponding at period "t-1" is the unlevered value of the company which, by definition, does not depend on the actual or posterior debt levels and hence, can be found as the present value of the future FCFs discounted at their respective Ku.

Using (6) with the example from table 12 we have

Table 18a Calculation of D_{Opt} and VL

Year	0	1	2	3	4
Ku		15.00%	15.00%	15.00%	15.00%
Kd		11.00%	11.00%	11.00%	11.00%
ψ		28.22%	29.92%	32.31%	36.45%
Ke		28.22%	29.92%	32.31%	36.45%
T		35.00%	35.00%	35.00%	35.00%
FCF		17.0000	20.0000	22.0000	25.0000
VU	58.6647	50.4644	38.0340	21.7391	0.0000
CFE		8.5342	8.2569	7.2180	5.3679
E	16.6724	12.8434	8.4286	3.9341	0.0000
TS		1.7340	1.5320	1.1895	0.7054
VTS	3.0463	2.1720	1.2897	0.5170	0.0000
D_{Opt} from (6)	45.0385	39.7930	30.8951	18.3221	
VL = E + D	61.7109	52.6364	39.3237	22.2561	0.0000
APV = VU + VTS	61.7109	52.6364	39.3237	22.2561	0.0000
D% = D_{Opt}/VL	72.9831%	75.5998%	78.5660%	82.3237%	
VU - D	13.6261	10.6714	7.1389	3.4171	

As can be seen the results are identical to the ones obtained using Solver. To illustrate the calculation of D_{Opt} we show for t -1 = 0 the value of debt in table 24b using eq. (6).

Table 18b. Calculation of components of Eq. (6)

$\dfrac{VU_{t-1}.(1+Ku_t)}{(1+Kd_t)}$	60.77871247
$\dfrac{(1+Kd_t)}{(1+Ku_t)}$	0.965217391
$\dfrac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}$	0.03344884
(6)	45.03854992

Now we illustrate the calculations for the two first periods of our example. Suppose that we wish to optimize the optimal capital structure of a company which future FCFs equal the first two years of the one shown in Table 1. The results, in this case will be:

Table 19. Application of formula (6)

Year	0	1	2
Ku		15.00%	15.00%
Kd		11.00%	11.00%
T		35.00%	35.00%
FCF		17.00	20.00
TS		0.93	0.56
VTS	1.02	0.41	0.00
CFE = FCF - CFD + TS		5.64	4.29
VU	29.91	17.39	0.00
(1) Market equity value	6.65	3.14	0.00
(2) Value of debt	24.28	14.66	0.00
(3) Ke =Ku + (Ku-Kd).D/(VU-D)		32.25%	36.45%
(4) VL	30.93	17.80	0.00

$$D_{OPT,1} = \frac{(17.39).(1+0.15)}{(1+0.11)}.\left\{1-\sqrt{1-\frac{(1+0.11)}{(1+0.15)}.\left[1-\frac{(0.00).(0.15-0.11)}{(17.39).(1+0.15).(0.11).(0.35)}\right]}\right\} = 14.66$$

$$D_{OPT,0} = \frac{(29.91).(1+0.15)}{(1+0.11)}.\left\{1-\sqrt{1-\frac{(1+0.11)}{(1+0.15)}.\left[1-\frac{(0.41).(0.15-0.11)}{(29.91).(1+0.15).(0.11).(0.35)}\right]}\right\} = 24.28$$

These results can be represented in a graphic as follows:

Exhibit 4: VL_0 for two periods as function of debt level at t=0 and t=1

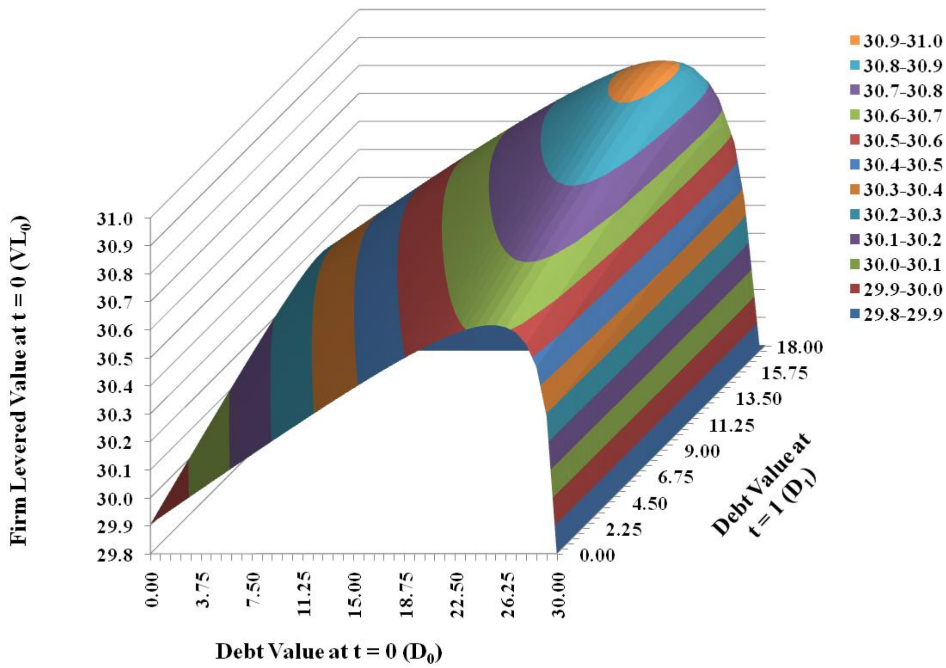
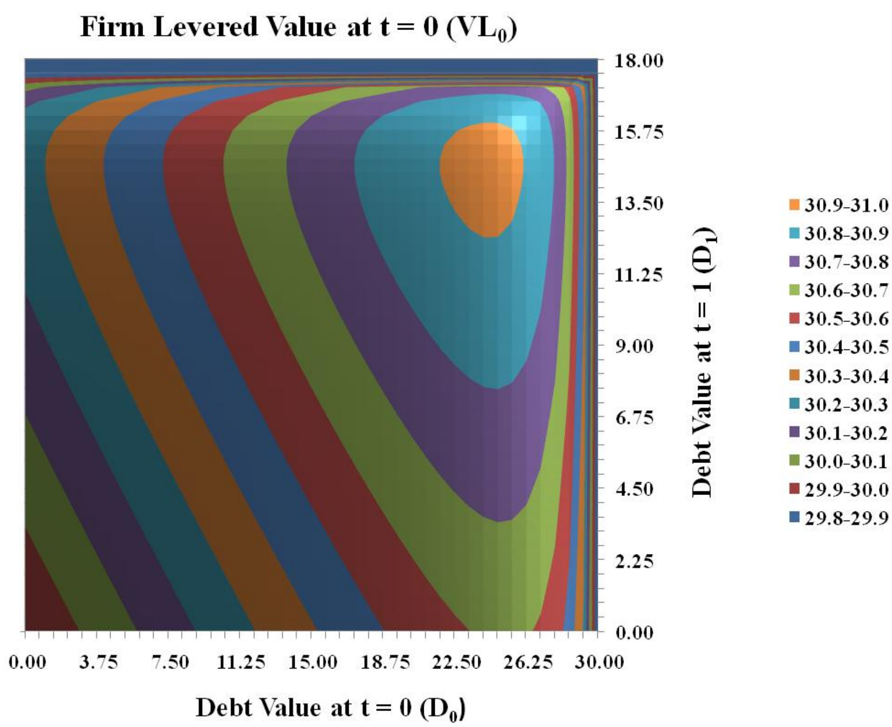


Exhibit 5: VL_0 for two periods as function of debt level at t=0 and t=1



Observe that formula (6) does not require any previous calculation of Ke, CFE or equity value. In consequence, the optimal level of debt resulting from its application can be plugged directly into expression (7) to find the optimal leverage without carrying the mentioned calculations (see Appendix A for details)

$$\mathbf{D\%} = \frac{\mathbf{D_{t-1}.[VU_{t-1}.(1+Ku_t) - D_{t-1}.(1+Kd_t)]}}{\mathbf{VU_{t-1}.[VU_{t-1}.(1+Ku_t) - D_{t-1}.(1+Kd_t)] + (VU_{t-1} - D_{t-1}).(VTS_t + D_{t-1}.Kd_t.T)}} \quad \mathbf{(7)}$$

Expression (7) is the same as (A40) in Appendix A and is used to graph VL as a function of leverage (see Figures 3 - 4):

Exhibit 6: VL_0 for two periods as function of leverage at t=0 and t=1

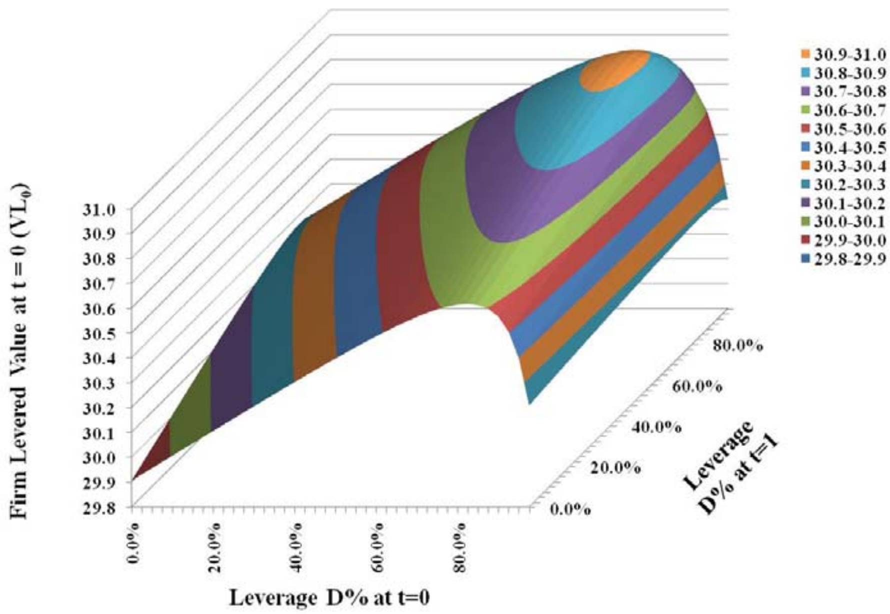
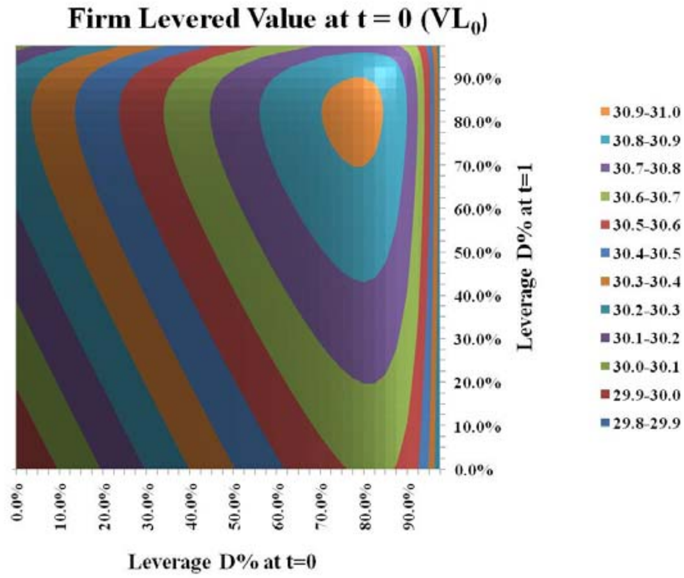


Exhibit 7: VL_0 for two periods as function of leverage at t=0 and t=1



The results obtained by means of (7) are D%=82.3237% for t=1 and a leverage of D%=78.5045% for t=0. It is crucial to note that the behavior observed in Exhibits 4 to 7 can be depicted graphically because they were constructed assuming the existence of only two periods. Thus, a similar procedure for a greater number of periods would require a higher-dimensional space to be visualized; nevertheless this number does not restrict the existence of an optimal capital structure and formula (6) still holds.

In order to complete the exposition, the case for optimal capital structure in perpetuities with constant growth is now presented. The formula for finding the optimal debt level is as follows (see Appendix B for details):

$$\mathbf{D_{Opt,t-1}} = \mathbf{VU_{t-1}}.\frac{(\mathbf{Ku_t - g})}{(\mathbf{Kd_t - g})}.\left[\mathbf{1} - \left[\mathbf{1} - \frac{(\mathbf{Kd_t - g})}{(\mathbf{Ku_t - g})}\right]^{1/2}\right] \qquad (\mathbf{8})$$

Formula (8) is the same as formula (B40) from Appendix B. The particular case of a non-growing perpetuity corresponds to g=0 and (8) collapses to

$$\mathbf{D_{Opt,t-1}} = \mathbf{VU_{t-1}}.\frac{\mathbf{Ku_t}}{\mathbf{Kd_t}}.\left[\mathbf{1} - \left[\mathbf{1} - \frac{\mathbf{Kd_t}}{\mathbf{Ku_t}}\right]^{1/2}\right] \qquad (\mathbf{9})$$

Formula (9) is the same formula (B41a) in Appendix B. Since in the mentioned case $VU_{t-1}=FCF_t/Ku_t$, expression (B41a) can be presented in the following equivalent form (which is the one derived by Tham et al, 2010) and can be found in Appendix B as (B41b):

$$\mathbf{D_{Opt,t-1}} = \frac{\mathbf{FCF_t}}{\mathbf{Kd_t}} \cdot \left[\mathbf{1} - \left[\mathbf{1} - \frac{\mathbf{Kd_t}}{\mathbf{Ku_t}}\right]^{1/2}\right] \qquad (\mathbf{10})$$

4. Concluding Remarks

This paper has shown the procedure to calculate value and optimal capital structure assuming Ke as the discount rate for TS for finite cash flows. Several scenarios were analyzed: constant leverage and variable leverage (D% different for each period), and closed-form analytical solutions for a finite number of periods and perpetuities. Five popular methods were used and all of them give the same identical answer.

Using an analytical formulation to calculate optimal debt, the differences in value are negligible. In this particular case, the differences between constant and variable leverage are very small: 0.009%; nonetheless, this paper does not claim that this difference is a general behavior when using one or other approach, but it remains clear that the variable leverage approach leads to equal or higher values of the levered company due to the value of the flexibility of adapting the leverage every period as a function of the expected future cash flows.

In addition, it was assumed that the cost of debt remains constant while changing the debt level in a particular period in order to maximize it. Nonetheless, more work has to be done in order to identify the behavior of Kd as a function of leverage. Consequently, our formulation is open to include a variable Kd, although linked to leverage which will create a circularity (observe that we are not referring to a Kd that is allowed to change from period to period because the herein derived formulas account for that kind of flexibility; we refer to the cost of debt in a particular period that could be allowed to change as a function of a variation in leverage).

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Appendix A

Derivation of the general formula for the optimal capital structure when there are a finite number of periods and ψ_t = Ke$_t$

We start from the following basic tenets of finance:

$$VL_{t-1} = E_{t-1} + D_{t-1} = VU_{t-1} + VTS_{t-1} \tag{A1}$$

$$TS_t = D_{t-1}.Kd_t.T \tag{A2}$$

$$VTS_t = VTS_{t-1}.(1+\psi_t) - TS_t \tag{A3}$$

$$VTS_{t-1} = \frac{VTS_t + TS_t}{1+\psi_t} \tag{A4}$$

And the expression for Ke as the discount rate for TS (See Tham, Velez-Pareja & Kolari 2010):

$$Ke_t = \psi_t = Ku_t + \frac{(Ku_t - Kd_t).D_{t-1}}{VU_{t-1} - D_{t-1}} \tag{A5}$$

The goal of this section is to find the value of D_{t-1} that maximizes the value of the levered firm, VL_{t-1}. It is necessary to note that is much easier to find the optimum using the absolute debt level D_{t-1} instead of the leverage D% = D_{t-1} / (E_{t-1} + D_{t-1}) since the latter yields more complex expressions. In consequence, the procedure consists of finding the derivative of VL_{t-1} with respect to D_{t-1}, equal it to zero, and solve the resulting equation with D_{t-1} as the unknown variable to find the optimum.

In order to find the mentioned derivative the chain rule of differential calculus is used due to the fact that the resulting expression will be a product of several factors, which makes easier to solve the equations for the optimal values (although, as will be seen, there is only one):

$$\frac{dVL_{t-1}}{dD_{t-1}} = \frac{dVL_{t-1}}{dVTS_{t-1}}.\frac{dVTS_{t-1}}{d\psi_t}.\frac{d\psi_t}{dD_{t-1}} \tag{A6}$$

Since expressions for each of the factors on the right hand side (RHS) of (A6) are needed, we proceed to find the third one in the first place by taking derivatives at both sides of (A5) with respect to D_{t-1}:

$$\frac{d\psi_t}{dD_{t-1}} = \frac{dKu_t}{dD_{t-1}} + \frac{(VU_{t-1} - D_{t-1}).\frac{d}{dD_{t-1}}[(Ku_t - Kd_t).D_{t-1}] - (Ku_t - Kd_t).D_{t-1}.\frac{d}{dD_{t-1}}(VU_{t-1} - D_{t-1})}{(VU_{t-1} - D_{t-1})^2}$$

$$\frac{d\psi_t}{dD_{t-1}} = 0 + \frac{(VU_{t-1} - D_{t-1}).(Ku_t - Kd_t) - (Ku_t - Kd_t).D_{t-1}.(-1)}{(VU_{t-1} - D_{t-1})^2}$$

$$\frac{d\psi_t}{dD_{t-1}} = \frac{(VU_{t-1} - D_{t-1}).(Ku_t - Kd_t) + (Ku_t - Kd_t).D_{t-1}}{(VU_{t-1} - D_{t-1})^2}$$

$$\frac{d\psi_t}{dD_{t-1}} = \frac{(VU_{t-1} - D_{t-1} + D_{t-1}).(Ku_t - Kd_t)}{(VU_{t-1} - D_{t-1})^2}$$

$$\frac{d\psi_t}{dD_{t-1}} = \frac{VU_{t-1}.(Ku_t - Kd_t)}{(VU_{t-1} - D_{t-1})^2} \tag{A7}$$

Note that VU_{t-1} and Ku_{t-1} are treated as constants since they are, by definition, independent of debt level. In contrast, Kd_t may increase with leverage, a situation that would demand an expression for the cost of debt as a function of $D_{t-1;}$ nonetheless, in this article Kd_t is assumed for simplicity considerations to be independent of debt level. Thus, taking the derivative of (A4) with respect to ψ_t, we have that

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(1+\psi_t).\frac{d}{d\psi_t}(VTS_t + TS_t) - (VTS_t + TS_t).\frac{d}{d\psi_t}(1+\psi_t)}{(1+\psi_t)^2} \tag{A8}$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(1+\psi_t).\frac{d}{d\psi_t}(VTS_t + D_{t-1}.Kd_t.T) - (VTS_t + D_{t-1}.Kd_t.T).\frac{d}{d\psi_t}(1+\psi_t)}{(1+\psi_t)^2} \quad (A9) = (A2) \text{ in } (A8)$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(1+\psi_t).Kd_t.T.\frac{dD_{t-1}}{d\psi_t} - (VTS_t + D_{t-1}.Kd_t.T).(1)}{(1+\psi_t)^2} \tag{A10}$$

$$\frac{dD_{t-1}}{d\psi_t} = \left(\frac{d\psi_t}{dD_{t-1}}\right)^{-1} \tag{A11}$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(1+\psi_t).Kd_t.T.\left[\frac{(VU_{t-1} - D_{t-1})^2}{VU_{t-1}.(Ku_t - Kd_t)}\right] - (VTS_t + D_{t-1}.Kd_t.T)}{(1+\psi_t)^2} \quad (A12) = (A7), (A11) \text{ in } (A10)$$

Observe that VTS_t is also treated as a constant, which is of crucial importance; this stems from the fact that it is a function of ψ_{t+1} and values of ψ of further periods (if they exist), but is independent of ψ_t. Nevertheless, this is not true in the case of TS_t, since it depends on D_{t-1} which, in turn, is related to ψ_t via (A5).

Now we proceed to find the derivative of the firm's levered value with respect to the present Value of Tax Shield (VTS) using (A1):

$$\frac{dVL_{t-1}}{dVTS_{t-1}} = \frac{dVU_{t-1}}{dVTS_{t-1}} + \frac{dVTS_{t-1}}{dVTS_{t-1}} \tag{A13}$$

Since the firm's unlevered value is independent of VTS,

$$\frac{dVL_{t-1}}{dVTS_{t-1}} = 0 + 1 = 1 \tag{A14}$$

Putting the factors (A14), (A12) and (A7) together in (A6) and setting the resultant derivative equal to zero, we have that

$$\frac{dVL_{t-1}}{dD_{t-1}} = (1).\left\{\frac{(1+\psi_t).Kd_t.T.\left[\frac{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)}{(1+\psi_t)^2}\right\}.\left[\frac{VU_{t-1}.(Ku_t-Kd_t)}{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}\right]=0 \tag{A15}$$

$$\frac{dVL_{t-1}}{dD_{t-1}} = \left\{\frac{(1+\psi_t).Kd_t.T.\left[\frac{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)}{(1+\psi_t)^2}\right\}.\left[\frac{VU_{t-1}.(Ku_t-Kd_t)}{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}\right] = 0 \tag{A16}$$

The solutions for the optimal debt level $D_{OPT,t-1}$ stemming from the first factor of (A16) are:

$$\frac{(1+\psi_t).Kd_t.T.\left[\frac{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)}{(1+\psi_t)^2}=0$$

$$(1+\psi_t).Kd_t.T.\left[\frac{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right) = 0 \tag{A17}$$

$$\left[1+Ku_t+\frac{(Ku_t-Kd_t).D_{OPT,t-1}}{VU_{t-1}-D_{OPT,t-1}}\right].Kd_t.T.\left[\frac{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)=0$$

$$(A18) = (A5) \text{ in } (A17)$$

$$\left[\frac{(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right)+(Ku_t-Kd_t).D_{OPT,t-1}}{VU_{t-1}-D_{OPT,t-1}}\right].Kd_t.T.\left[\frac{\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}\right] - \left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)=0$$

$$\left[(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right)+(Ku_t-Kd_t).D_{OPT,t-1}\right].Kd_t.T.\left[\frac{VU_{t-1}-D_{OPT,t-1}}{VU_{t-1}.(Ku_t-Kd_t)}\right] - \left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)=0$$

$$(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right).Kd_t.T.\left[\frac{VU_{t-1}-D_{OPT,t-1}}{VU_{t-1}.(Ku_t-Kd_t)}\right] + (Ku_t-Kd_t).D_{OPT,t-1}.Kd_t.T.\left[\frac{VU_{t-1}-D_{OPT,t-1}}{VU_{t-1}.(Ku_t-Kd_t)}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right)=0$$

$$\frac{(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right)^2.Kd_t.T}{VU_{t-1}.(Ku_t-Kd_t)}+D_{OPT,t-1}.Kd_t.T.\left[\frac{VU_{t-1}-D_{OPT,t-1}}{VU_{t-1}}\right]-\left(VTS_t+D_{OPT,t-1}.Kd_t.T\right) = 0$$

$$\frac{(1+Ku_t).Kd_t.T.\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}+D_{OPT,t-1}.Kd_t.T.\left[\frac{VU_{t-1}-D_{OPT,t-1}}{VU_{t-1}}-1\right]-VTS_t=0$$

$$\frac{(1+Ku_t).Kd_t.T.\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}+D_{OPT,t-1}.Kd_t.T.\left[\frac{VU_{t-1}-D_{OPT,t-1}-VU_{t-1}}{VU_{t-1}}\right]-VTS_t=0$$

$$\frac{(1+Ku_t).Kd_t.T.\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}-\frac{D^2_{OPT,t-1}.Kd_t.T}{VU_{t-1}}-VTS_t=0$$

$$\frac{(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{VU_{t-1}.(Ku_t-Kd_t)}-\frac{D^2_{OPT,t-1}}{VU_{t-1}}-\frac{VTS_t}{Kd_t.T}=0$$

$$\frac{(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right)^2}{(Ku_t-Kd_t)}-D^2_{OPT,t-1}-\frac{VU_{t-1}.VTS_t}{Kd_t.T}=0$$

$$(1+Ku_t).\left(VU_{t-1}-D_{OPT,t-1}\right)^2-D^2_{OPT,t-1}.(Ku_t-Kd_t)-\frac{VU_{t-1}.(Ku_t-Kd_t).VTS_t}{Kd_t.T}=0$$

$$(1+Ku_t).\left(VU^2_{t-1}-2.VU_{t-1}.D_{OPT,t-1}+D^2_{OPT,t-1}\right)-D^2_{OPT,t-1}.(Ku_t-Kd_t)-\frac{VU_{t-1}.(Ku_t-Kd_t).VTS_t}{Kd_t.T}=0$$

$$-D^2_{OPT,t-1}.(Ku_t-Kd_t-1-Ku_t)-2.D_{OPT,t-1}.VU_{t-1}.(1+Ku_t)-\frac{VU_{t-1}.(Ku_t-Kd_t).VTS_t}{Kd_t.T} + VU^2_{t-1}.(1+Ku_t)=0$$

$$-D^2_{OPT,t-1}.(1+Kd_t)-2.D_{OPT,t-1}.VU_{t-1}.(1+Ku_t)+\frac{VU^2_{t-1}.(1+Ku_t).Kd_t.T-VU_{t-1}.(Ku_t-Kd_t).VTS_t}{Kd_t.T} = 0$$

$$D^2_{OPT,t-1}.(1+Kd_t)-2.D_{OPT,t-1}.VU_{t-1}.(1+Ku_t)+\frac{VU_{t-1}.\left[VU_{t-1}.(1+Ku_t).Kd_t.T-(Ku_t-Kd_t).VTS_t\right]}{Kd_t.T} = 0$$

$$\mathbf{D^2_{OPT,t-1}.\frac{(1+Kd_t)}{VU_{t-1}}-2.D_{OPT,t-1}.(1+Ku_t)+\frac{VU_{t-1}.(1+Ku_t).Kd_t.T-(Ku_t-Kd_t).VTS_t}{Kd_t.T}=0} \quad (\mathbf{A19})$$

This is a quadratic equation in $D_{OPT,t-1}$ that can be solved by means of the general formula:

$$\mathbf{A = \frac{(1+Kd_t)}{VU_{t-1}}} \tag{A20}$$

$$\mathbf{B = -2.(1+Ku_t)} \tag{A21}$$

$$\mathbf{C = \frac{VU_{t-1}.(1+Ku_t).Kd_t.T-(Ku_t-Kd_t).VTS_t}{Kd_t.T}} \tag{A22}$$

$$\mathbf{D_{OPT,t-1} = \frac{-B \pm \sqrt{B^2-4.A.C}}{2.A}} \tag{A23}$$

$$\mathbf{D_{OPT,t-1} = \frac{2.(1+Ku_t) \pm \sqrt{4.(1+Ku_t)^2-4.\frac{(1+Kd_t)}{VU_{t-1}}.\frac{VU_{t-1}.(1+Ku_t).Kd_t.T-(Ku_t-Kd_t).VTS_t}{Kd_t.T}}}{2.\frac{(1+Kd_t)}{VU_{t-1}}}}$$

(A24) = (A20), (A21) and (A22) in (A23)

$$D_{OPT,t-1} = \frac{(1+Ku_t) \pm \sqrt{(1+Ku_t)^2-\frac{(1+Kd_t)}{VU_{t-1}}.\frac{VU_{t-1}.(1+Ku_t).Kd_t.T-(Ku_t-Kd_t).VTS_t}{Kd_t.T}}}{\frac{(1+Kd_t)}{VU_{t-1}}}$$

$$D_{OPT,t-1} = \frac{(1+Ku_t) \pm \sqrt{(1+Ku_t)^2-\frac{VU_{t-1}.(1+Ku_t).(1+Kd_t).Kd_t.T-(Ku_t-Kd_t).(1+Kd_t).VTS_t}{VU_{t-1}.Kd_t.T}}}{\frac{(1+Kd_t)}{VU_{t-1}}}$$

$$D_{OPT,t-1} = \frac{(1+Ku_t) \pm \sqrt{(1+Ku_t)^2-(1+Ku_t).(1+Kd_t)+\frac{(Ku_t-Kd_t).(1+Kd_t).VTS_t}{VU_{t-1}.Kd_t.T}}}{\frac{(1+Kd_t)}{VU_{t-1}}}$$

$$D_{OPT,t-1} = \frac{VU_{t-1}}{(1+Kd_t)}.\left[(1+Ku_t) \pm \sqrt{(1+Ku_t)^2-(1+Ku_t).(1+Kd_t)+\frac{(Ku_t-Kd_t).(1+Kd_t).VTS_t}{VU_{t-1}.Kd_t.T}}\right]$$

$$D_{OPT,t-1} = \frac{VU_{t-1}.(1+Ku_t)}{(1+Kd_t)}.\left[1 \pm \sqrt{\frac{(1+Ku_t)^2}{(1+Ku_t)^2}-\frac{(1+Ku_t).(1+Kd_t)}{(1+Ku_t)^2}+\frac{(Ku_t-Kd_t).(1+Kd_t).VTS_t}{VU_{t-1}.(1+Ku_t)^2.Kd_t.T}}\right]$$

$$D_{OPT,t-1} = \frac{VU_{t-1}.(1+Ku_t)}{(1+Kd_t)}.\left[1 \pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}+\frac{(1+Kd_t)}{(1+Ku_t)}\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}}\right]$$

$$\mathbf{D_{OPT,t-1} = \frac{VU_{t-1}.(1+Ku_t)}{(1+Kd_t)}.\left\{1 \pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]}\right\}} \tag{A25}$$

Equation (A25) has one root for a positive sign before the radical, and one for the negative sign; thus, we need to identify which one is consistent from a financial point of view:

$$\frac{D_{OPT,t-1}}{VU_{t-1}} = \frac{(1+Ku_t)}{(1+Kd_t)}.\left\{1 \pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]}\right\} \quad \text{(A26)}$$

Kolari and Velez-Pareja (2010) show that VUt-1>Dt-1; otherwise, it would imply a negative value for the unlevered equity. In consequence:

$$\frac{D_{OPT,t-1}}{VU_{t-1}} < 1 \quad \text{(A27)}$$

$$\frac{(1+Ku_t)}{(1+Kd_t)}.\left\{1 \pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]}\right\} < 1 \quad \text{(A28)} = \text{(A26) in (A27)}$$

$$1 \pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]} < \frac{(1+Kd_t)}{(1+Ku_t)}$$

$$\pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]} < \frac{(1+Kd_t)}{(1+Ku_t)} - 1 \quad \text{(A29)}$$

In addition, Ku_t must be greater than Kd_t, by definition:

$$\frac{(1+Kd_t)}{(1+Ku_t)} - 1 < 0 \quad \text{(A30)}$$

$$\pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]} < \frac{(1+Kd_t)}{(1+Ku_t)} - 1 < 0 \quad \text{(A31)} = \text{(A30) and (A29)}$$

$$\pm \sqrt{1-\frac{(1+Kd_t)}{(1+Ku_t)}.\left[1-\frac{VTS_t.(Ku_t-Kd_t)}{VU_{t-1}.(1+Ku_t).Kd_t.T}\right]} < 0 \quad \text{(A32)}$$

In order to yield real roots, the expression under the radical must be positive and, consequently, (A32) holds only if the sign before the radical is negative:

$$-\sqrt{\mathbf{1}-\frac{(\mathbf{1}+\mathbf{Kd_t})}{(\mathbf{1}+\mathbf{Ku_t})}.\left[\mathbf{1}-\frac{\mathbf{VTS_t}.(\mathbf{Ku_t}-\mathbf{Kd_t})}{\mathbf{VU_{t-1}}.(\mathbf{1}+\mathbf{Ku_t}).\mathbf{Kd_t}.\mathbf{T}}\right]} < 0 \quad \textbf{(A33)}$$

Hence, the formula for the optimal debt level in a specific period "t-1" is

$$\mathbf{D_{OPT,t-1}} = \frac{\mathbf{VU_{t-1}}.(\mathbf{1}+\mathbf{Ku_t})}{(\mathbf{1}+\mathbf{Kd_t})}.\left\{\mathbf{1} - \sqrt{\mathbf{1}-\frac{(\mathbf{1}+\mathbf{Kd_t})}{(\mathbf{1}+\mathbf{Ku_t})}.\left[\mathbf{1}-\frac{\mathbf{VTS_t}.(\mathbf{Ku_t}-\mathbf{Kd_t})}{\mathbf{VU_{t-1}}.(\mathbf{1}+\mathbf{Ku_t}).\mathbf{Kd_t}.\mathbf{T}}\right]}\right\} \quad \textbf{(A34)}$$

And, finally, for the second factor of (A16) we have that

$$\frac{\mathrm{VU}_{t-1}.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{(\mathrm{VU}_{t-1} - \mathrm{D}_{t-1})^2} = 0$$
$$\mathrm{VU}_{t-1}.(\mathrm{Ku}_t - \mathrm{Kd}_t) = 0$$

The former equation yields no solutions since its left hand side (LHS) is independent of the variable D_{t-1}. Going back to (A34), the following analysis looks for the conditions needed to obtain real roots from that equation; that is, under what conditions the expression under the radical is equal or greater than zero:

$$1 - \frac{(1+\mathrm{Kd}_t)}{(1+\mathrm{Ku}_t)}.\left[1 - \frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}}\right] \geq 0$$
$$\frac{(1+\mathrm{Kd}_t)}{(1+\mathrm{Ku}_t)}.\left[1 - \frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}}\right] \leq 1$$
$$1 - \frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}} \leq \frac{(1+\mathrm{Ku}_t)}{(1+\mathrm{Kd}_t)}$$
$$-\frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}} \leq \frac{(1+\mathrm{Ku}_t)}{(1+\mathrm{Kd}_t)} - 1$$
$$1 - \frac{(1+\mathrm{Ku}_t)}{(1+\mathrm{Kd}_t)} \leq \frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}}$$
$$\frac{(1+\mathrm{Kd}_t) - (1+\mathrm{Ku}_t)}{(1+\mathrm{Kd}_t)} \leq \frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}}$$
$$-\frac{(\mathrm{Ku}_t - \mathrm{Kd}_t)}{(1+\mathrm{Kd}_t)} \leq \frac{\mathrm{VTS}_t.(\mathrm{Ku}_t - \mathrm{Kd}_t)}{\mathrm{VU}_{t-1}.(1+\mathrm{Ku}_t).\mathrm{Kd}_t.\mathrm{T}}$$
$$\frac{\mathbf{VTS_t}.(\mathbf{1}+\mathbf{Kd_t})}{\mathbf{VU_{t-1}}.(\mathbf{1}+\mathbf{Ku_t}).\mathbf{Kd_t}.\mathbf{T}} \geq -\mathbf{1} \qquad (\mathbf{A35})$$

Since (1 + Kdt), (1 + Kut), and Kdt.T must be positive, the only cases for which an optimal capital structure would not exist are those where the quotient (VTSt / VUt-1) is negative. This situation may be observed in firms with a limited duration such as concessions, in which the exploited asset (a highway, for instance) must be transferred to a third party (a local municipality, for instance) in optimal conditions at the end of the concession and, thus, demand an important investment in the last period. This could cause the FCF of that period to be negative and, consequently, also VU would be negative in the previous period.

Up to this point, an expression for the optimal debt level (A34) has been found, but it is still lacked a formula for the optimal leverage. In consequence, the following derivation presents an expression for the leverage D% in terms of D_{t-1}, Kd_t, VU_{t-1}, Ku_t and VTS_t:

$$VL_{t-1} = E_{t-1} + D_{t-1} = VU_{t-1} + VTS_{t-1} \tag{A1}$$

$$VTS_{t-1} = \frac{VTS_t + TS_t}{1 + \psi_t} \tag{A4}$$

$$TS_t = D_{t-1}.Kd_t.T \tag{A2}$$

$$\psi_t = Ku_t + \frac{(Ku_t - Kd_t).D_{t-1}}{VU_{t-1} - D_{t-1}} \tag{A5}$$

$$\psi_t = \frac{Ku_t.(VU_{t-1} - D_{t-1}) + (Ku_t - Kd_t).D_{t-1}}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{Ku_t.VU_{t-1} - Ku_t.D_{t-1} + Ku_t.D_{t-1} - Kd_t.D_{t-1}}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{VU_{t-1}.Ku_t - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}} \tag{A36}$$

$$D\% = \frac{D_{t-1}}{VL_{t-1}} \tag{A37}$$

$$VTS_{t-1} = \frac{VTS_t + D_{t-1}.Kd_t.T}{1 + \frac{VU_{t-1}.Ku_t - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}}} \qquad (A38) = (A36) \text{ in } (A4)$$

$$VTS_{t-1} = \frac{VTS_t + D_{t-1}.Kd_t.T}{\frac{VU_{t-1} - D_{t-1} + VU_{t-1}.Ku_t - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}}}$$

$$VTS_{t-1} = \frac{(VU_{t-1} - D_{t-1}).(VTS_t + D_{t-1}.Kd_t.T)}{VU_{t-1}.(1 + Ku_t) - D_{t-1}.(1 + Kd_t)}$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \frac{(VU_{t-1} - D_{t-1}).(VTS_t + D_{t-1}.Kd_t.T)}{VU_{t-1}.(1 + Ku_t) - D_{t-1}.(1 + Kd_t)}} \qquad (A39)\ (A38) \text{ in } (A37)$$

$$\mathbf{D\% = \frac{D_{t-1}.[VU_{t-1}.(1 + Ku_t) - D_{t-1}.(1 + Kd_t)]}{VU_{t-1}.[VU_{t-1}.(1 + Ku_t) - D_{t-1}.(1 + Kd_t)] + (VU_{t-1} - D_{t-1}).(VTS_t + D_{t-1}.Kd_t.T)}} \tag{A40}$$

In consequence, by means of (A40) and using $D_{OPT,t-1}$ as the value for D_{t-1}, it is possible to find the optimal value for the leverage. Now, as a final step, we proceed to derive an expression for D_{t-1} as a function of leverage, a result that can be used to plot VL as a function of leverage:

$$D_{t-1}.[VU_{t-1}.(1+Ku_t) - D_{t-1}.(1+Kd_t)]$$
$$= D\%.VU_{t-1}.[VU_{t-1}.(1+Ku_t) - D_{t-1}.(1+Kd_t)]$$
$$+ D\%.(VU_{t-1} - D_{t-1}).(VTS_t + D_{t-1}.Kd_t.T)$$

$$D_{t-1}.VU_{t-1}.(1+Ku_t) - D_{t-1}^2.(1+Kd_t)$$
$$= D\%.VU_{t-1}^2.(1+Ku_t) - D\%.VU_{t-1}.D_{t-1}.(1+Kd_t)$$
$$+ (D\%.VU_{t-1} - D\%.D_{t-1}).(VTS_t + D_{t-1}.Kd_t.T)$$

$$D_{t-1}.VU_{t-1}.(1+Ku_t) - D_{t-1}^2.(1+Kd_t)$$
$$= D\%.VU_{t-1}^2.(1+Ku_t) - D\%.VU_{t-1}.D_{t-1}.(1+Kd_t) + D\%.VU_{t-1}.VTS_t$$
$$+ D\%.VU_{t-1}.D_{t-1}.Kd_t.T - D\%.D_{t-1}.VTS_t - D\%.D_{t-1}^2.Kd_t.T$$

$$-D_{t-1}^2.(1+Kd_t - D\%.Kd_t.T)$$
$$+ D_{t-1}.[VU_{t-1}.(1+Ku_t) + D\%.VU_{t-1}.(1+Kd_t) - D\%.VU_{t-1}.Kd_t.T + D\%.VTS_t]$$
$$- D\%.VU_{t-1}^2.(1+Ku_t) - D\%.VU_{t-1}.VTS_t = 0$$

$$-D_{t-1}^2.[1+Kd_t.(1-D\%.T)] + D_{t-1}.\{VU_{t-1}.(1+Ku_t) + D\%.VU_{t-1}.[1+Kd_t.(1-T)] + D\%.VTS_t\}$$
$$- D\%.VU_{t-1}.[VU_{t-1}.(1+Ku_t) + VTS_t] = 0$$

$$\mathbf{D_{t-1}^2.[1+Kd_t.(1-D\%.T)] - D_{t-1}.\{VU_{t-1}.(1+Ku_t) + D\%.VU_{t-1}.[1+Kd_t.(1-T)] + D\%.VTS_t\}}$$
$$\mathbf{+ D\%.VU_{t-1}.[VU_{t-1}.(1+Ku_t) + VTS_t] = 0} \qquad \mathbf{(A41)}$$

Equation (A41) has a quadratic form and, thus, can be solved by means of the general formula

$$\mathbf{D_{t-1} = \frac{-B \pm \sqrt{B^2 - 4.A.C}}{2.A}} \qquad \mathbf{(A23)}$$

And the following values for A,B and C:

$$\mathbf{A = 1 + Kd_t.(1 - D\%.T)} \qquad \mathbf{(A42)}$$
$$\mathbf{B = -VU_{t-1}.(1+Ku_t) - D\%.\{VU_{t-1}.[1+Kd_t.(1-T)] + VTS_t\}} \qquad \mathbf{(A43)}$$
$$\mathbf{C = D\%.VU_{t-1}.[VU_{t-1}.(1+Ku_t) + VTS_t]} \qquad \mathbf{(A44)}$$

The full resulting expression is not shown in this document due to length considerations, but its numerical application is very straightforward.

Appendix B

Derivation of the general formula for the optimal capital structure for perpetuities with constant growth when $\psi_t = Ke_t$

The problem of deriving a general recursive formula for finding the optimal capital structure in every period of a valuation process with a finite number of them was solved in Appendix A. Thus, Appendix B focuses in solving the same problem for perpetuities, a process that is very similar to that presented in the previous appendix. Hence, the mentioned derivation starts from the following basic tenets of finance for perpetuities:

$$E_{t-1} = \frac{CFE_t}{Ke_t - g} \qquad (B1)$$

$$VU_{t-1} = \frac{FCF_t}{Ku_t - g} \qquad (B2)$$

$$VTS_{t-1} = \frac{TS_t}{\psi_t - g} \qquad (B3)$$

$$VL_{t-1} = E_{t-1} + D_{t-1} = VU_{t-1} + VTS_{t-1} \qquad (B4)$$

$$E_{t-1} + D_{t-1} = \frac{FCF_t}{Ku_t - g} + \frac{TS_t}{\psi_t - g} \qquad (B5) = (B2), (B3) \text{ in } (B4)$$

$$TS_t = D_{t-1}.Kd_t.T \qquad (A2)$$

$$CFE_t = FCF_t + D_t - D_{t-1} - D_{t-1}Kd_t.(1 - T) \qquad (B6)$$

$$D_t = D_{t-1}.(1 + g) \qquad (B7)$$

$$D_t - D_{t-1} = D_{t-1}.g \qquad (B8)$$

$$CFE_t = FCF_t + D_{t-1}.g - D_{t-1}.Kd_t + D_{t-1}.Kd_t.T \qquad (B9) = (B8) \text{ in } (B6)$$

$$CFE_t = FCF_t - D_{t-1}.(Kd_t - g) + TS_t \qquad (B10) = (B6) \text{ in } (B9)$$

$$\psi_t = Ke_t \qquad (B11)$$

$$\frac{CFE_t}{\psi_t - g} + D_{t-1} = VU_{t-1} + \frac{TS_t}{\psi_t - g} \qquad (B12) = (B1), (B2), (B3) \text{ and } (B11) \text{ in } (B4)$$

$$\frac{CFE_t - TS_t}{\psi_t - g} = VU_{t-1} - D_{t-1} \qquad (B13)$$

$$\frac{FCF_t - D_{t-1}.(Kd_t - g) + TS_t - TS_t}{\psi_t - g} = VU_{t-1} - D_{t-1} \qquad (B14) = (B10) \text{ in } (B13)$$

$$\frac{FCF_t - D_{t-1}.(Kd_t - g)}{\psi_t - g} = VU_{t-1} - D_{t-1}$$

$$\psi_t = \frac{FCF_t - D_{t-1}.(Kd_t - g)}{VU_{t-1} - D_{t-1}} + g$$

$$\psi_t = \frac{FCF_t - D_{t-1}.(Kd_t - g) + g.(VU_{t-1} - D_{t-1})}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{FCF_t - D_{t-1}.Kd_t + D_{t-1}.g + VU_{t-1}.g - D_{t-1}.g}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{FCF_t - D_{t-1}.Kd_t + VU_{t-1}.g}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{\frac{FCF_t.(Ku_t - g)}{Ku_t - g} - D_{t-1}.Kd_t + \frac{FCF_t}{Ku_t - g}.g}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{\frac{FCF_t.(Ku_t - g + g)}{Ku_t - g} - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{\frac{FCF_t.(Ku_t)}{Ku_t - g} - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}}$$

$$\boldsymbol{\psi_t = \frac{VU_{t-1}.Ku_t - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}}} \qquad (\mathbf{B16})$$

$$\psi_t = \frac{VU_{t-1}.Ku_t - Ku_t.D_{t-1} + Ku_t.D_{t-1} - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}}$$

$$\psi_t = \frac{Ku_t.(VU_{t-1} - D_{t-1}) + (Ku_t - Kd_t).D_{t-1}}{VU_{t-1} - D_{t-1}}$$

$$\boldsymbol{\psi_t = Ku_t + \frac{(Ku_t - Kd_t).D_{t-1}}{VU_{t-1} - D_{t-1}}} \qquad (\mathbf{B17})$$

Using the chain rule of differential calculus, we have:

$$\frac{dVL_{t-1}}{dD_{t-1}} = \frac{dVL_{t-1}}{dVTS_{t-1}}.\frac{dVTS_{t-1}}{d\psi_t}.\frac{d\psi_t}{dD_{t-1}} \qquad (B18)$$

We now find expressions for every individual derivative in then LHS of (B18):

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{d}{d\psi_t}\frac{TS_t}{\psi_t - g} = \frac{d}{d\psi_t}\frac{D_{t-1}.Kd_t.T}{\psi_t - g}$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(\psi_t - g).\frac{d}{d\psi_t}D_{t-1}.Kd_t.T - D_{t-1}.Kd_t.T.\frac{d}{d\psi_t}(\psi_t - g)}{(\psi_t - g)^2}$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(\psi_t - g).Kd_t.T.\frac{dD_{t-1}}{d\psi_t} - D_{t-1}.Kd_t.T.\frac{d\psi_t}{d\psi_t}}{(\psi_t - g)^2}$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{(\psi_t - g).Kd_t.T.\frac{dD_{t-1}}{d\psi_t} - D_{t-1}.Kd_t.T.\frac{d\psi_t}{d\psi_t}}{(\psi_t - g)^2}$$

$$\frac{dVTS_{t-1}}{d\psi_t} = \frac{Kd_t.T.\left[(\psi_t - g)\frac{dD_{t-1}}{d\psi_t} - D_{t-1}\right]}{(\psi_t - g)^2} \qquad (B19)$$

$$\frac{dD_{t-1}}{d\psi_t}=\left(\frac{d\psi_t}{dD_{t-1}}\right)^{-1} \tag{A11}$$

$$\frac{dD_{t-1}}{d\psi_t}=\left(\frac{d\psi_t}{dD_{t-1}}\right)^{-1}=\frac{(VU_{t-1}-D_{t-1})^2}{VU_{t-1}.(Ku_t-Kd_t)} \qquad \text{(B20) = (A7) in (A11)}$$

$$\frac{dVTS_{t-1}}{d\psi_t}=\frac{Kd_t.T.\left[\frac{(\psi_t-g).(VU_{t-1}-D_{t-1})^2}{VU_{t-1}.(Ku_t-Kd_t)}-D_{t-1}\right]}{(\psi_t-g)^2} \qquad \text{(B21) = (B20) in (B19)}$$

$$\frac{dVTS_{t-1}}{d\psi_t}=\frac{Kd_t.T.\left[\frac{(\psi_t-g).(VU_{t-1}-D_{t-1})^2-D_{t-1}.VU_{t-1}.(Ku_t-Kd_t)}{VU_{t-1}.(Ku_t-Kd_t)}\right]}{(\psi_t-g)^2}$$

$$\boldsymbol{\frac{dVTS_{t-1}}{d\psi_t}=\frac{Kd_t.T.\left[(\psi_t-g).(VU_{t-1}-D_{t-1})^2-D_{t-1}.VU_{t-1}.(Ku_t-Kd_t)\right]}{VU_{t-1}.(Ku_t-Kd_t).(\psi_t-g)^2}} \tag{\textbf{B22}}$$

$$\frac{dVL_{t-1}}{dVTS_{t-1}}=\frac{dVU_{t-1}}{dVTS_{t-1}}+\frac{dVTS_{t-1}}{dVTS_{t-1}} \tag{B23}$$

$$\frac{dVL_{t-1}}{dVTS_{t-1}}=0+1=1 \tag{B24}$$

$$\frac{dVL_{t-1}}{dD_{t-1}}=(1).\left[\frac{Kd_t.T.\left[(\psi_t-g).(VU_{t-1}-D_{t-1})^2-D_{t-1}.VU_{t-1}.(Ku_t-Kd_t)\right]}{VU_{t-1}.(Ku_t-Kd_t).(\psi_t-g)^2}\right].\left[\frac{VU_{t-1}.(Ku_t-Kd_t)}{(VU_{t-1}-D_{t-1})^2}\right]$$

(B25) = (B24), (B22), (A7) in (B18)

Now we equal the derivative of the levered value of the firm with respect to the debt level to zero in order to find the optimum:

$$\boldsymbol{\frac{dVL_{t-1}}{dD_{t-1}}=\left[\frac{Kd_t.T.\left[(\psi_t-g).(VU_{t-1}-D_{OPT,t-1})^2-D_{OPT,t-1}.VU_{t-1}.(Ku_t-Kd_t)\right]}{VU_{t-1}.(Ku_t-Kd_t).(\psi_t-g)^2}\right].\left[\frac{VU_{t-1}.(Ku_t-Kd_t)}{(VU_{t-1}-D_{OPT,t-1})^2}\right]=0}$$

(**B25**)

First, we look for solutions stemming from the first factor of the LHS of (B25):

$$\frac{Kd_t.T.\left[(\psi_t-g).(VU_{t-1}-D_{OPT,t-1})^2-D_{OPT,t-1}.VU_{t-1}.(Ku_t-Kd_t)\right]}{VU_{t-1}.(Ku_t-Kd_t).(\psi_t-g)^2}=0$$

$$\left[Ku_t+\frac{(Ku_t-Kd_t).D_{OPT,t-1}}{VU_{t-1}-D_{t-1}}-g\right].(VU_{t-1}-D_{OPT,t-1})^2-D_{OPT,t-1}.VU_{t-1}.(Ku_t-Kd_t)=0$$

$$\left[\frac{(Ku_t-g).(VU_{t-1}-D_{OPT,t-1})+(Ku_t-Kd_t).D_{OPT,t-1}}{VU_{t-1}-D_{OPT,t-1}}\right].(VU_{t-1}-D_{OPT,t-1})^2$$
$$-D_{OPT,t-1}.VU_{t-1}.(Ku_t-Kd_t)=0$$

$$\left[\frac{VU_{t-1}.(Ku_t - g) + (Ku_t - Kd_t - Ku_t + g).D_{OPT,t-1}}{VU_{t-1} - D_{OPT,t-1}}\right].\left(VU_{t-1} - D_{OPT,t-1}\right)^2 - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) = 0$$

$$\left[\frac{FCF_t - (Kd_t - g).D_{OPT,t-1}}{VU_{t-1} - D_{OPT,t-1}}\right].\left(VU_{t-1} - D_{OPT,t-1}\right)^2 - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) = 0$$

$$\left[FCF_t - (Kd_t - g).D_{OPT,t-1}\right].\left(VU_{t-1} - D_{OPT,t-1}\right) - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) = 0$$

$$VU_{t-1}.\left[FCF_t - (Kd_t - g).D_{OPT,t-1}\right] - D_{OPT,t-1}.\left[FCF_t - (Kd_t - g).D_{OPT,t-1}\right] - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) = 0$$

$$VU_{t-1}.\left[FCF_t - (Kd_t - g).D_{OPT,t-1}\right] - D_{OPT,t-1}.\left[FCF_t - (Kd_t - g).D_{t-1}\right] - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) = 0$$

$$VU_{t-1}.FCF_t - VU_{t-1}.(Kd_t - g).D_{OPT,t-1} - D_{OPT,t-1}.FCF_t + (Kd_t - g).D_{OPT,t-1}^2 - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) = 0$$

$$(Kd_t - g).D_{OPT,t-1}^2 - VU_{t-1}.(Kd_t - g).D_{OPT,t-1} - D_{OPT,t-1}.VU_{t-1}.(Ku_t - Kd_t) - D_{OPT,t-1}.FCF_t + VU_{t-1}.FCF_t = 0$$

$$(Kd_t - g).D_{OPT,t-1}^2 - [VU_{t-1}.(Kd_t - g) + VU_{t-1}.(Ku_t - Kd_t) + FCF_t].D_{OPT,t-1} + VU_{t-1}.FCF_t = 0$$

$$(Kd_t - g).D_{OPT,t-1}^2 - \{VU_{t-1}[Kd_t - g + Ku_t - Kd_t] + FCF_t\}.D_{OPT,t-1} + VU_{t-1}.FCF_t = 0$$

$$(Kd_t - g).D_{OPT,t-1}^2 - [VU_{t-1}.(Ku_t - g) + FCF_t].D_{OPT,t-1} + VU_{t-1}.FCF_t = 0$$

$$(Kd_t - g).D_{OPT,t-1}^2 - [FCF_t + FCF_t].D_{OPT,t-1} + VU_{t-1}.FCF_t = 0$$

$$(Kd_t - g).D_{OPT,t-1}^2 - 2.FCF_t.D_{OPT,t-1} + VU_{t-1}.FCF_t = 0$$

$$\frac{(Kd_t - g)}{FCF_t}.D_{OPT,t-1}^2 - 2.D_{OPT,t-1} + VU_{t-1} = 0 \qquad \text{(B26)}$$

Equation (B26) has a quadratic form with $D_{OPT,t-1}$ as the unknown; it is now solved using the corresponding general formula:

$$\mathbf{A = \frac{(Kd_t - g)}{FCF_t}} \qquad \mathbf{(B27)}$$

$$\mathbf{B = -2} \qquad \mathbf{(B28)}$$

$$\mathbf{C = VU_{t-1}} \qquad \mathbf{(B29)}$$

$$\mathbf{D_{OPT,t-1} = \frac{-B \pm \sqrt{B^2 - 4.A.C}}{2.A}} \qquad \mathbf{(A23)}$$

$$D_{OPT,t-1} = \frac{2 \pm \sqrt{4 - 4.\frac{VU_{t-1}.(Kd_t - g)}{FCF_t}}}{2.\frac{(Kd_t - g)}{FCF_t}} \qquad \text{(B30)} = \text{(B27)}, \text{(B28)}, \text{(B29) in (A23)}$$

$$D_{OPT,t-1} = \frac{1 \pm \sqrt{1 - \frac{VU_{t-1}.(Kd_t - g)}{FCF_t}}}{\frac{(Kd_t - g)}{FCF_t}}$$

$$D_{OPT,t-1} = \frac{FCF_t}{(Kd_t - g)}.\left[1 \pm \sqrt{1 - \frac{FCF_t.(Kd_t - g)}{FCF_t.(Ku_t - g)}}\right] \tag{B31}$$

$$\mathbf{D_{OPT,t-1} = VU_{t-1}.\frac{(Ku_t - g)}{(Kd_t - g)}.\left[1 \pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}}\right]} \tag{\textbf{B32}}$$

The observation made in Appendix A regarding the fact that when $Ke_t=\psi_t$ the debt level must be inferior to the unlevered value of the company, is now used to find whether the sign behind the radical should be positive or negative:

$$\frac{D_{OPT,t-1}}{VU_{t-1}} < 1 \tag{B33}$$

$$\frac{D_{OPT,t-1}}{VU_{t-1}} = \frac{(Ku_t - g)}{(Kd_t - g)}.\left[1 \pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}}\right] \tag{B34}$$

$$\frac{(Ku_t - g)}{(Kd_t - g)}.\left[1 \pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}}\right] < 1 \tag{B35) = (B34) in (B33}$$

$$1 \pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}} < \frac{(Kd_t - g)}{(Ku_t - g)} \tag{B36}$$

Since Ku_t should be greater than Kd_t,

$$\frac{(Kd_t - g)}{(Ku_t - g)} < 1 \tag{B37}$$

$$1 \pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}} < \frac{(Kd_t - g)}{(Ku_t - g)} < 1 \tag{B38) = (B37) and (B36}$$

$$1 \pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}} < 1$$

$$\pm \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}} < 0 \tag{B39}$$

Since the expression under the radical should be positive since $Ku_t>Kd_t$, inequality (B39) only holds if the sign before the radical is negative:

$$-\sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}} < 0 \tag{B39a}$$

In consequence, the general formula for the optimal debt value in a perpetuity with constant growth is:

$$\mathbf{D_{OPT,t-1} = VU_{t-1}.\frac{(Ku_t - g)}{(Kd_t - g)}.\left[1 - \sqrt{1 - \frac{(Kd_t - g)}{(Ku_t - g)}}\right]} \qquad (\mathbf{B40})$$

In the particular case of a non-growing perpetuity, g=0 and (B40) collapses to

$$\mathbf{D_{OPT,t-1} = VU_{t-1}.\frac{Ku_t}{Kd_t}.\left[1 - \sqrt{1 - \frac{Kd_t}{Ku_t}}\right]} \qquad (\mathbf{B41a})$$

Since in the mentioned case $VU_{t-1}=FCF_t/Ku_t$, expression (B41a) can be presented in the following equivalent form (which is the one derived by Tham et al, 2010):

$$\mathbf{D_{OPT,t-1} = \frac{FCF_t}{Kd_t}.\left[1 - \sqrt{1 - \frac{Kd_t}{Ku_t}}\right]} \qquad (\mathbf{B41b})$$

In the next step, expressions for D% as a function of any value of D_{t-1} in the context of growing perpetuities are derived:

$$D\% = \frac{D_{t-1}}{E_{t-1} + D_{t-1}} = \frac{D_{t-1}}{VU_{t-1} + VTS_{t-1}} \qquad (B42)$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \dfrac{D_{t-1}.Kd_t.T}{\psi_t - g}} \qquad (B43) = (B3), (A2) \text{ in } (B42)$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \dfrac{D_{t-1}.Kd_t.T}{\dfrac{VU_{t-1}.Ku_t - D_{t-1}.Kd_t}{VU_{t-1} - D_{t-1}} - g}} \qquad (B44) = (B16) \text{ in } (B43)$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \dfrac{D_{t-1}.Kd_t.T}{\dfrac{VU_{t-1}.Ku_t - D_{t-1}.Kd_t - (VU_{t-1} - D_{t-1}).g}{VU_{t-1} - D_{t-1}}}}$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \dfrac{D_{t-1}.Kd_t.T}{\dfrac{VU_{t-1}.Ku_t - D_{t-1}.Kd_t - VU_{t-1}.g + D_{t-1}.g}{VU_{t-1} - D_{t-1}}}}$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \dfrac{D_{t-1}.Kd_t.T}{\dfrac{VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)}{VU_{t-1} - D_{t-1}}}}$$

$$D\% = \frac{D_{t-1}}{VU_{t-1} + \dfrac{D_{t-1}.Kd_t.T.(VU_{t-1} - D_{t-1})}{VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)}}$$

$$D\% = \frac{D_{t-1}}{\dfrac{VU_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)] + D_{t-1}.Kd_t.T.(VU_{t-1} - D_{t-1})}{VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)}}$$

$$D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)] + D_{t-1}.Kd_t.T.(VU_{t-1} - D_{t-1})}$$

$$D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}^2.(Ku_t - g) - VU_{t-1}.D_{t-1}.(Kd_t - g) + VU_{t-1}.D_{t-1}.Kd_t.T - D_{t-1}^2.Kd_t.T}$$

$$D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}^2.(Ku_t - g) - VU_{t-1}.D_{t-1}.(Kd_t - g - Kd_t.T) - D_{t-1}^2.Kd_t.T}$$

$$\mathbf{D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}^2.(Ku_t - g) - VU_{t-1}.D_{t-1}.[Kd_t.(1 - T) - g] - D_{t-1}^2.Kd_t.T}} \qquad (\mathbf{B45a})$$

Or, equivalently,

$$D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}.FCF_t - VU_{t-1}.D_{t-1}.[Kd_t.(1 - T) - g] - D_{t-1}^2.Kd_t.T} \qquad (B45b) = (B2) \text{ in } (B45a)$$

$$\mathbf{D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}.\{FCF_t - D_{t-1}.[Kd_t.(1 - T) - g]\} - D_{t-1}^2.Kd_t.T}} \qquad (\mathbf{B45c})$$

For non-growing perpetuities, g=0; thus,

$$\mathbf{D\% = \frac{D_{t-1}.[VU_{t-1}.Ku_t - D_{t-1}.Kd_t]}{VU_{t-1}^2.Ku_t - VU_{t-1}.D_{t-1}.Kd_t.(1 - T) - D_{t-1}^2.Kd_t.T}} \qquad (\mathbf{B46a})$$

Or equivalently,

$$D\% = \frac{D_{t-1}.[FCF_t - D_{t-1}.Kd_t]}{VU_{t-1}.FCF_t - VU_{t-1}.D_{t-1}.Kd_t.(1 - T) - D_{t-1}^2.Kd_t.T} \qquad (B46b)$$

$$\mathbf{D\% = \frac{D_{t-1}.[FCF_t - D_{t-1}.Kd_t]}{VU_{t-1}.[FCF_t - D_{t-1}.Kd_t.(1 - T)] - D_{t-1}^2.Kd_t.T}} \qquad (\mathbf{B46c})$$

As was mentioned previously in Appendix A, the numerical application of these formulas with $D_{t-1}=D_{Opt,t-1}$ make possible the calculation of the optimal leverage. Finally, a plot of VL_{t-1} as a function of leverage is presented in the main part of this document, in which the following formulation is used:

$$D\% = \frac{D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]}{VU_{t-1}^2.(Ku_t - g) - VU_{t-1}.D_{t-1}.[Kd_t.(1 - T) - g] - D_{t-1}^2.Kd_t.T}$$

$$D\%.\{VU_{t-1}^2.(Ku_t - g) - VU_{t-1}.D_{t-1}.[Kd_t.(1 - T) - g] - D_{t-1}^2.Kd_t.T\}$$
$$= D_{t-1}.[VU_{t-1}.(Ku_t - g) - D_{t-1}.(Kd_t - g)]$$

$$D_{t-1}^2.[D\%.Kd_t.T - (Kd_t - g)] + D_{t-1}.\{VU_{t-1}.(Ku_t - g) + VU_{t-1}.D\%.[Kd_t.(1 - T) - g]\}$$
$$- VU_{t-1}^2.D\%.(Ku_t - g) = 0$$

$$D_{t-1}^2.[D\%.Kd_t.T - (Kd_t - g)] + D_{t-1}.VU_{t-1}.[Ku_t - g + D\%.Kd_t.(1 - T) - D\%.g]$$
$$- VU_{t-1}^2.D\%.(Ku_t - g) = 0$$

$$\mathbf{D_{t-1}^2.\frac{[D\%.Kd_t.T-(Kd_t-g)]}{VU_{t-1}}+D_{t-1}.[Ku_t-g.(1+D\%)+D\%.Kd_t.(1-T)]-VU_{t-1}.D\%.(Ku_t-g)=0} \quad \mathbf{(B47)}$$

Equation (B47) has a quadratic form with D_{t-1} as the unknown variable and, thus, can be solved by means of the general formula

$$\mathbf{D_{t-1}=\frac{-B\pm\sqrt{B^2-4.A.C}}{2.A}} \quad \mathbf{(A23)}$$

with the values for A, B and C presented next:

$$\mathbf{A=\frac{[D\%.Kd_t.T-(Kd_t-g)]}{VU_{t-1}}} \quad \mathbf{(B48)}$$

$$\mathbf{B=[Ku_t-g.(1+D\%)+D\%.Kd_t.(1-T)]} \quad \mathbf{(B49)}$$

$$\mathbf{C=-VU_{t-1}.D\%.(Ku_t-g)} \quad \mathbf{(B50)}$$

The full resulting expression for D_{t-1} is not shown in this document due to length considerations, but its numerical application is very straightforward.